Argumentieren und Beweisen SS 2017

Exercise Sheet 1: Proof Structure

Some of the exercises and solutions have been taken from Wohlgemuth's book.

Exercise 1.

Translate each of the following statements into a first-order formula using $\in$ as the only predicate symbol:

1.1 If $A \subseteq B$, then A and $C \setminus B$ are disjoint.

1.2 A is not a subset of B.

1.3 $\neg\big(A \cup B \subseteq C \setminus D\big)$.

Exercise 2.

Eliminate def^2 from the proof fragment below.

1.	$a \in M$	
2.	$M \subseteq N$	
3.	$a \in N$	(1,2; def$^2\subseteq$)

Exercise 3.

Let A and B be sets. Define $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$ and $\overline{B} = \{x \in U \mid x \notin B\}$, where U is the universal set. Prove $(A \setminus B) = (A \cap \overline{B})$ in the following two steps.

3.1 Show: $(A \setminus B) \subseteq (A \cap \overline{B})$.

3.2 Show: $(A \cap \overline{B}) \subseteq (A \setminus B)$.

Exercise 4.

Show: $\forall$ sets A $\forall$ sets B: $((A \setminus B) \cup B) = (A \cup B)$. Contrary to the examples in the lecture, handle the forall statements on the object level and not on the meta level.

Exercise 5.

Show: $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$ holds for sets $A_1, \ldots, A_n$.

Exercise 6.

Assume that A, B, and C are sets. Show: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 7.
For sets A, B, and C:

7.1 Show: $A \subseteq B$ iff $A \cap B = A$.

7.2 Show: $A \subseteq B$ iff $A \cup B = B$.

Exercise 8.
Given sets A_k (for $k = 1, 2, 3, 4$), define

$$B_1 = A_1 \qquad B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus (A_1 \cup A_2) \qquad B_4 = A_4 \setminus (A_1 \cup A_2 \cup A_3)\,.$$

8.1 Prove: $\bigcup_{i=1}^{4} B_i = \bigcup_{i=1}^{4} A_i$

8.2 Prove: For all $1 \leq i, j \leq 4, i \neq j$: $B_i \cap B_j = \emptyset$

Exercise 9.
Let $f\colon A \mapsto B$. Suppose $X \subseteq A$, $Y \subseteq B$, and $y \in B$. Define:

P1 $f(X) = \{b \in B \mid b = f(x) \text{ for some } x \in X\}$

P2 $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$

P3 $f^{-1}(y) = \{a \in A \mid f(a) = y\}$

Let $C \subseteq B$, and $D \subseteq B$. Show: $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. First structure the proof in the same way as we disussed in the lecture. Then present the proof in English.

Exercise 10.
Prove or refute the following:
Let $f\colon A \mapsto B$, $E \subseteq A$, $F \subseteq A$. Then $f(E \cap F) = f(E) \cap f(F)$.

Exercise 11.
A function $f\colon A \mapsto B$ is called *one-to-one* provided that for all $a_1, a_2 \in A$: if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be defined by $f(x) = 2x + 4$ for all $x \in \mathbb{R}$. Prove that f is one-to-one.

Exercise 12.
A function $f\colon A \mapsto B$ is called *onto* if for each $b \in B$ there exists some $a \in A$ such that $f(a) = b$.
Show that there exists a one-to-one function $g\colon \mathbb{N} \mapsto \mathbb{N}$ which is not onto.

Exercise 13.

A *sequence* is a function $a\colon \mathbb{N} \mapsto \mathbb{R}$. $a(n)$ is called the nth term of the sequence and is denoted by a_n. The sequence itself is denoted by $\langle a_1, a_2, \ldots \rangle$ or by $\langle a_n \rangle$. A number $L \in \mathbb{R}$ is defined to be the *limit* of $\langle a_n \rangle$, written $\lim_{n\to\infty}\langle a_n \rangle = L$, provided that, given a real number $\epsilon > 0$, there exists a natural number N such that for all $n > N\colon |a_n - L| < \epsilon$. If L is the limit of $\langle a_n \rangle$, then $\langle a_n \rangle$ *converges* to L. A sequence with no limit is said to *diverge*.

Show that $\langle a_n \rangle$ defined by $a_n = n/(n+1)$ has limit 1.

Exercise 14.

Let $\langle a_n \rangle$ be a sequence. If $\lim_{n\to\infty}\langle a_n \rangle = L$ and $\lim_{n\to\infty}\langle a_n \rangle = M$, then $L = M$. That is, if a sequence has a limit, then it is unique.

Hint: $|x - z| \le |x - y| + |y - z|$ for all $x, y, z \in \mathbb{R}$.