Argumentieren und Beweisen SS 2019

Exercise Sheet (Induction)

This exercise sheet provides some training material for the optional tutorial.

Exercise 1.

Show that any natural number above 11 can be written as a sum of 4's and 5's.

Exercise 2.

Show that for any $n \in \mathbb{N}_0$, the number $\ell_n = 4^{n+2} + 5^{2n+1}$ is divisible by 21.

Exercise 3.

Prove the following: *If a tree has $n \geq 1$ vertices, then it has $n - 1$ edges.*

Exercise 4.

Let I, I' be interpretations with domains $\mathcal{U}$, $\mathcal{U}'$. We say that I is *isomorphic* to I' if there exists a bijection $\chi : \mathcal{U} \to \mathcal{U}'$ such that the following conditions are fulfilled:

1. $\chi(I(c)) = I'(c)$ for every constant symbol c.
2. $\chi(I(f)(p_1, \ldots, p_n)) = I'(f)(\chi(p_1), \ldots, \chi(p_n))$ for every n-ary function symbol f ($n > 0$) and all $p_1, \ldots, p_n \in \mathcal{U}$.

χ is called *isomorphism* between I and I'.

Now let χ be an isomorphism between I and I'. Prove that for all closed (i.e., variable-free) terms t we have $I'(t) = \chi(I(t))$.

Hint: Prove the statement by structural induction on t.

Recall the following definitions:

- *Closed term*: Each constant symbol is a closed term. If f is an n-ary function symbol and $t_1, \ldots t_n$ are closed terms then $f(t_1, \ldots, t_n)$ is a closed term.
- *Interpretation of a term*: Given an interpretation I for all constants and function symbols. The interpretation I is extended to closed terms inductively as follows. For an arbitrary n-ary function symbol I and terms $t_1, \ldots t_n$:

$$I\left(f(t_1, \ldots, t_n)\right) = I(f)\left(I(t_1), \ldots, I(t_n)\right)$$

Exercise 5.

We define the set L of all lists as follows:

$$L ::= nil \mid (c : L)$$

nil denotes the empty list containing no element. We define the function *append* by $append(nil, y) = y$ and $append((c : x), y) = (c : append(x, y))$. Show that, for all lists ℓ, $append(\ell, nil) = \ell$ holds.

Exercise 6.
Recall from former lectures the definition of *Fibonacci numbers*: $F(0) = 0$, $F(1) = 1$, and for all integers $k \geq 0$, $F(k+2) = F(k+1) + F(k)$. Show that, for all integers $n \geq 0$, $F(n) < 2^n$ holds.

Exercise 7.
Recall from former lectures the definition of *Fibonacci numbers*: $F(0) = 0$, $F(1) = 1$, and for all integers $k \geq 0$, $F(k+2) = F(k+1) + F(k)$. Show that, for all integers $n \geq 0$, $\sum_{i=0}^{n-1} F(i) < F(n+1)$ holds.

Exercise 8.
Recall from former lectures the definition of *Fibonacci numbers*: $F(0) = 0$, $F(1) = 1$, and for all integers $k \geq 0$, $F(k+2) = F(k+1) + F(k)$. Show that, for all integers $n \geq 3$, $\alpha^{n-2} < F(n)$ holds, where $\alpha = (1 + \sqrt{5})/2$.
Remark: α is called the *golden ratio* and it can be proved that $\lim_{n\to\infty} F_{n+1}/F_n = \alpha$.

Exercise 9.
What is wrong with the following proof of the statement: *for any positive real x and any natural number n, $x^n = 1$ holds*?

Let $\mathcal{P}(n)$ denote $x^n = 1$. The proof is by mathematical induction on n.

Base case. For $n = 0$, $\mathcal{P}(n)$ is true because $x^0 = 1$.

Induction hypothesis. Assume $\mathcal{P}(n)$ is true for some $n \geq 0$.

Induction step. We want to show that $\mathcal{P}(n+1)$ is true. We derive:

$$\begin{aligned} x^{n+1} &= \frac{x^n \cdot x^n}{x^{n-1}} \\ &= \frac{1 \cdot 1}{1} \qquad \text{(from above by the induction hypothesis)} \\ &= 1\,. \end{aligned}$$

Hence, $\mathcal{P}(n+1)$ is true. □

Exercise 10.
Let $\mathcal{P}(n)$ denote the statement $n! > 2^n$. Show that there is a smallest natural number n_0 such that $\mathcal{P}(n)$ holds for all natural numbers $n \geq n_0$.

Exercise 11.
For sets $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$, define $B_1 = A_1$ and $B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i)$ for $n > 1$.

Prove: $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$.

Exercise 12.
Let us define the *Fermat* numbers $F_n = 2^{(2^n)} + 1$ for all natural numbers $n \geq 1$. Prove that for all $n \geq 1$, $F_n = (F_0 \cdot F_1 \cdot \cdots \cdot F_{n-1}) + 2$.

Exercise 13.
Let $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Then $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$.

Exercise 14.
Suppose we have an operation (denoted by $'$) which is applied to symbols. The operation $'$ satisfies the following three axioms:

$$(u+v)' = u' + v' \qquad (A+')$$
$$(uv)' = uv' + u'v \qquad (A\cdot')$$
$$(cu)' = cu' \qquad (Ac')$$

Define $w^{(k)}$ inductively as follows:

B: $w^{(0)} = w$

S: Suppose $w^{(n)}$ is already defined for some natural number $n \geq 0$. Then $w^{(n+1)} = \left(w^{(n)}\right)'$.

Prove Leibniz's formula $(uv)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(i)} v^{(n-i)}$. (Recall that $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$.)

Exercise 15.
This exercise is taken from Hopcroft, J.E., Motwani, R. and Ullman, J.D.: *Introduction to Automata Theory, Languages, and Computation*, 3rd ed., Pearson, 2007. It requires some background from automata theory.
A non-deterministic finite automaton A is a quintuple $(Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite set of input symbols, $q_0 \in Q$ is the start state, and $F \subseteq Q$ is the set of final (or accepting) states. The transition function δ takes a state from Q and an input symbol from Σ as arguments and returns a subset of Q.

The set of all strings over Σ is denoted by Σ^*. Let x and y be two strings. Then xy is the concatenation of them. The length of a string w (i.e., the number of symbol occurrences in w) is denoted by $|w|$. ϵ is the empty string which is of length 0.

We denote by $\hat{\delta}$ the extension of δ to strings.

$$\hat{\delta}(q, w) = \begin{cases} \{q\} & \text{if } w = \epsilon \text{ (the empty string);} \\ \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a) & \text{if } w = xa,\ x \in \Sigma^* \text{ and } a \in \Sigma. \end{cases}$$

The language accepted by A, $L(A)$, is $\{w \mid \hat{\delta}(q_0, w) \cap F \neq \{\}\}$.

Let $A = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$, where $\delta(q_0, 0) = \{q_0, q_1\}$, $\delta(q_0, 1) = \{q_0\}$, $\delta(q_1, 0) = \{\}$, $\delta(q_1, 1) = \{q_2\}$, and $\delta(q_2, 0) = \delta(q_2, 1) = \{\}$. Show that $L(A) = \{w \mid w \text{ ends in } 01\}$.

Hint. Use mutual induction on the following statements:

1. $\hat{\delta}(q_0, w)$ contains q_0 for every w.
2. $\hat{\delta}(q_0, w)$ contains q_1 if and only if w ends in 0.
3. $\hat{\delta}(q_0, w)$ contains q_2 if and only if w ends in 01.

Exercise 16.
Consider a (generalized) chess board of size $2^n \times 2^n$, where one position is cut out. Take an L-tile made of three positions and show for all natural numbers $n \geq 1$ that the chess board can be covered using the L-tiles. Compute the number of required L-tiles.

Exercise 17.
You have a bag with red, yellow and blue chips. If only one chip remains in the bag, you put it out. Otherwise you remove two chips at random:

1. If one of the removed chips is red, you do not put any chips in the bag.
2. If both of the removed chips are yellow, you put one yellow chip and five blue chips in the bag.
3. If one of the chips is blue and the other is not red, you put ten red chips in the bag.

Show that any sequence of moves applied to an arbitrary bag always terminates or provide a non-terminating sequence of moves.

Exercise 18.
This exercise requires some background from linear algebra, which we take from https://en.wikipedia.org/wiki/Matrix_exponential.

A *matrix exponential* is a matrix function defined as follows

$$e^M = \sum_{k=1}^{\infty} \frac{1}{k!} M^k = I + M + \frac{M^2}{2!} + \ldots$$

where M is an $n \times n$ (real or complex) matrix and $M^0 = I$ is the *identity* matrix. The series e^M *converges absolutely* (see http://math.ucr.edu/~res/math138A/expmatrix.pdf for the definition and a proof). Absolute convergence implies that manipulations on infinite sums like the Cauchy product

$$\Big(\sum_{i=0}^{\infty} A_i\Big)\Big(\sum_{j=0}^{\infty} B_j\Big) = \Big(\sum_{i,j=0}^{\infty} A_i B_j\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} A_k B_{n-k}\Big)$$

can be safely performed.

Show the following:

1. If $AB = BA$ then $A^k e^B = e^B A^k$ and $e^A e^B = e^B e^A$ for arbitrary $n \times n$ (real or complex) matrices and $k \in \mathbb{N}_0$.

2. $e^{(s+t)A} = e^{sA}e^{tA}$ for arbitrary $n \times n$ (real or complex) matrix A and (real or complex) numbers s and t. What can you say about the inverse of e^A?

Exercise 19.

From M. Sipser: Introduction to the theory of computation, third edition.

Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first n natural numbers and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = (S(n))^2$ for every n.

1. $S(n) = \frac{1}{2}n(n+1)$.
2. $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

Exercise 20. *Equivalent Replacement Theorem*

Prove the following statement:

> Let φ, ψ_1, and ψ_2 be propositional formulas. If $\psi_1 \equiv \psi_2$, then $\varphi[\psi_1] \equiv \varphi[\psi_2]$.

Remark about binomial coefficients We consider properties of binomial coefficients in the following. Here are some basic facts and definitions about them.

1. We write binomial coefficients in the form $\binom{a}{b}$, where a and b are non-negative integers.
2. For convenience we allow that b is an integer.
3. $\binom{a}{b}$ is calculated as follows: (i) If $a < b$ or $b < 0$ then $\binom{a}{b} = 0$ by definition. Otherwise $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.
4. $\binom{a}{b} = \binom{a}{a-b}$ and $\binom{a}{0} = \binom{a}{a} = 1$.
5. We will often use that $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$ for any integer n with $n \geq 0$ and integer r with $0 \leq r \leq n$.

Exercise 21.

Show for all non-negative integers n that the following holds:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Exercise 22.
Show for all positive integers n that the following holds:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Exercise 23.
Show for all non-negative integers n and m that the following holds:

$$\sum_{k=0}^{n} k\binom{m+k}{m} = n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2}.$$

Exercise 24.
Show for all integers $n > 0$ that the alternating sum of the binomial coefficients is zero, i.e., show that

$$\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n} = 0.$$

What can you say about the sub-sum of all binomial coefficients $\binom{n}{\ell}$ with $\ell \leq n$ and ℓ even and the similar sub-sum with ℓ odd?

Exercise 25.
Show for all integers $n \geq 1$ and all $\ell \geq 1$ that

$$\sum_{i=0}^{\ell}\binom{n+i}{i} = \binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+\ell}{\ell} = \binom{n+\ell+1}{\ell}.$$

Hint: Use mathematical induction on ℓ.

Exercise 26.
Show that, for all non-negative integers m, k, and n, the following holds:

$$\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$$

What can you say about the value of $\binom{2n}{n}$?
Hint: Use mathematical induction on m.

Exercise 27.
For positive integers m, n, and p, show that the following holds:

$$\binom{m+n}{m+p} = \sum_{k=0}^{k} \binom{m}{k} \binom{n}{p+k}$$

Exercise 28.
For all non-negative integers n and all real numbers x with $x > -1$, $(1+x)^n \geq 1 + nx$ holds.