# 6.0/4.0 VU Formale Methoden der Informatik (185.291)

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| Kennzahl (study id) | Matrikelnummer (student id) | Familienname (family name) | Vorname (first name) | Gruppe (version) |
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**1.)** Consider the following problem:

> **GRAPH-FORMULA (GF)**
>
> INSTANCE: A pair $(G, \varphi)$, where $G$ is an undirected graph and $\varphi$ is a propositional formula.
>
> QUESTION: Is it the case that $G$ is 3-colorable or $\varphi$ is satisfiable?

By providing a reduction from an NP-complete problem, prove that **GF** is an NP-hard problem. Argue formally that your reduction is correct. **(15 points)**

**2.)** (a) Apply the Sparse Method including preprocessing to the $E$-formula $\varphi$ to obtain an equi-satisfiable formula $\psi$ in propositional logic.

$$\varphi : \qquad \big[(x_1 \doteq x_5 \wedge x_5 \doteq x_3) \vee (x_1 \not\doteq x_2 \wedge x_2 \doteq x_3)\big] \wedge (x_5 \doteq x_4 \rightarrow x_4 \not\doteq x_3)$$

Please indicate and justify briefly the steps in the translation! **(6 points)**

(b) Recall that resolution is defined as follows: given two clauses

$$C_1 = (A_1 \vee \ldots \vee A_i \vee \ldots \vee A_n) \quad \text{and} \quad C_2 = (B_1 \vee \ldots \vee B_j \vee \ldots \vee B_m)$$

such that, for some $i$ with $1 \leq i \leq n$, $A_i = \neg B_j$, the resolvent of $C_1$ and $C_2$ on $A_i$ is the clause

$$res(C_1, C_2, A_i) = (A_1 \vee \ldots \vee A_{i-1} \vee A_{i+1} \vee \ldots \vee A_n \vee B_1 \ldots \vee B_{j-1} \vee B_{j+1} \vee \ldots \vee B_m).$$

Now let $F$ be a set of clauses and a let $F' = F \cup \{res(C_1, C_2, A_i)\}$ be the extension of $F$ by a resolvent of some clauses $C_1, C_2 \in F$ where $A_i$ is a literal occurring positively in $C_1$ and negatively in $C_2$.

**Prove:** If $F$ is valid, then $F'$ is valid.

**(9 points)**

**3.)** Consider the following modified if-rule:

$$\frac{\{\,F\,\}\,p\,\{\,G\,\} \quad \{\,F \wedge \neg e\,\}\,q\,\{\,G\,\}}{\{\,F\,\}\ \mathsf{if}\ e\ \mathsf{then}\ p\ \mathsf{else}\ q\ \mathsf{fi}\ \{\,G\,\}}\ \text{if}'$$

(a) Show that this rule is admissible regarding partial correctness. **(5 points)**

(b) Show that the Hoare calculus for partial correctness is no longer complete, if we replace the regular if-rule by the modified one. **(10 points)**

A rule $\frac{X_1 \cdots X_n}{\{F\}\,p\,\{G\}}$ is *admissible regarding partial correctness*, if the conclusion $\{F\}\,p\,\{G\}$ is partially correct whenever all premises $X_1, \ldots, X_n$ are valid formulas/partially correct assertions.

Hoare calculus for partial correctness:

$$\{F\}\,\mathsf{skip}\,\{F\} \qquad \{F\}\,\mathsf{abort}\,\{G\} \qquad \{F[v/e]\}\,v := e\,\{F\}$$

$$\frac{\{F\}\,p\,\{G\} \quad \{G\}\,q\,\{H\}}{\{F\}\,p;q\,\{H\}}$$

$$\frac{\{F \wedge e\}\,p\,\{G\} \quad \{F \wedge \neg e\}\,q\,\{G\}}{\{F\}\,\mathsf{if}\ e\ \mathsf{then}\ p\ \mathsf{else}\ q\ \mathsf{fi}\,\{G\}}$$

$$\frac{\{Inv \wedge e\}\,p\,\{Inv\}}{\{Inv\}\,\mathsf{while}\ e\ \mathsf{do}\ p\ \mathsf{od}\,\{Inv \wedge \neg e\}}$$

$$\frac{F \to F' \quad \{F'\}\,p\,\{G'\} \quad G' \to G}{\{F\}\,p\,\{G\}}$$

## 4.) Simulation

Let $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ be two Kripke structures.

**Simulation**

Remember, a relation $H \subseteq S_1 \times S_2$ is a simulation relation if for each $(s, s') \in H$ holds:

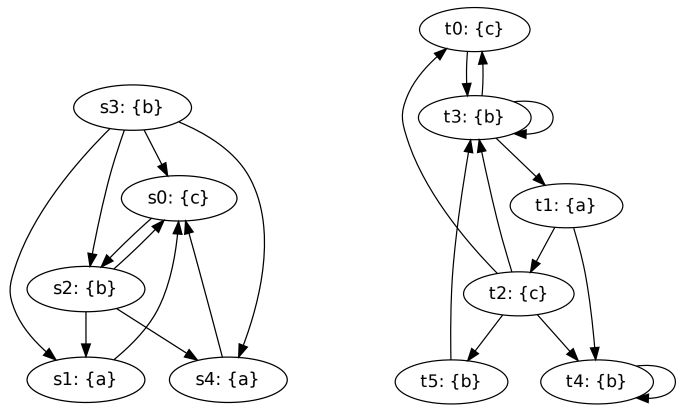
- $L_1(s) = L_2(s')$, and
- for each $(s, t) \in R_1$ there is a $(s', t') \in R_2$ such that $(t, t') \in H$.

Further remember, $M_2$ *simulates* $M_1$, in signs $M_1 \leq M_2$, if there is a simulation relation $H \subseteq S_1 \times S_2$ *such that*

- for each initial state $s \in I_1$ there is an initial state $s' \in I_2$ with $(s, s') \in H$.

In the following, we say that $H$ *witnesses the similarity of* $M_1$ *and* $M_2$ in case $H$ is a simulation relation from $M_1$ to $M_2$ that satisfies the condition stated above.

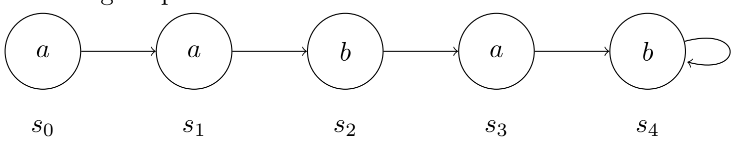
(a) Provide a non-empty simulation relation $H$ that witnesses $M_1 \leq M_2$, where $M_1$ and $M_2$ are shown below ($M_1$ on the left, $M_2$ on the right), the initial state of $M_1$ is $s_0$, the initial state of $M_2$ is $t_0$:




**(3 points)**

(b) We consider an extension of LTL with a yesterday operator $\mathbf{Y}$ where $\mathbf{Y}\phi$ is true if and only if $\phi$ was true in the previous state. Moreover, in the first state of a path, $\mathbf{Y}\phi$ is always false.

Consider the following Kripke structure:




- Determine on which states $s_i$ the following formulae hold:
  i. **FY**a
  ii. **YF**a

iii. $\mathbf{G}(\mathbf{Y}\mathrm{a} \rightarrow \mathrm{b})$
iv. $\mathbf{G}(\mathrm{a} \rightarrow \mathbf{Y}\mathrm{b})$

- Give an equivalent LTL formula (i.e. without $\mathbf{Y}$) for formula (iv), i.e., $\mathbf{G}(\mathrm{a} \rightarrow \mathbf{Y}\mathrm{b})$.

**(5 points)**

(c) Let $M = (S, R, L)$ be a Kripke structure. We define a Kripke structure $M' = (S', R', L')$ by

- $S' = \{(s, i) \mid s \in S \text{ and } i \in \{1, 2, 3\}\}$, i.e, for every state $s$ in $S$ there are three states $(s, 1), (s, 2), (s, 3)$ in $S'$,
- $R' = \{((s, 1), (s, 2)) \mid s \in S\} \cup \{((s, 2), (s, 3)) \mid s \in S\} \cup \{((s, 3), (t, 1)) \mid (s, t) \in R\}$, i.e., for every state $s \in S$ there are edges from $(s, 1)$ to $(s, 2)$ and from $(s, 2)$ to $(s, 3)$ in $R'$, and for every edge $(s, t)$ in $R$ there is an edge from $(s, 3)$ to $(t, 1)$ in $R'$,
- $L'((s, i)) = L(s)$ for all $i \in \{1, 2, 3\}$ and all $s \in S$, i.e., the label of state $(s, i)$ in $M'$ agrees with the label of state $s$ in $M$.

Show that for all states $s \in S$, all $i \in \{1, 2, 3\}$ and all LTL formulae $\phi$ without the $\mathbf{X}$ operator, it holds that $M, s \models \phi$ iff $M', (s, i) \models \phi$.
*Hint: You may want to use induction on the structure of the formula $\phi$.* **(7 points)**