6.0/4.0 VU Formale Methoden der Informatik 185.291 30 January 2015				
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1.) Consider the following 2 problems:

> **3-COLORABILITY (3-COL)**
>
> INSTANCE: An undirected graph $G = (V, E)$.
>
> QUESTION: Does G have a *3-coloring*? That is, does there exist a function μ from vertices in V to values in $\{1, 2, 3\}$ such that $\mu(v_1) \neq \mu(v_2)$ for any edge $[v_1, v_2] \in E$.

> **UNDIRECTED GRAPH HOMOMORPHISM (HOM)**
>
> INSTANCE: A pair (G_1, G_2), where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are undirected graphs.
>
> QUESTION: Does there exist a homomorphism from G_1 to G_2? That is, does there exist a function h from vertices in V_1 to vertices in V_2 such that: for any edge $[v_1, v_2] \in E_1$ we also have $[h(v_1), h(v_2)] \in E_2$?

We provide next a reduction from **3-COL** to **HOM**. Let $G = (V, E)$ be an arbitrary undirected graph (i.e. an arbitrary instance of **3-COL**). From G we construct a pair (G_1, G_2) of undirected graphs. We let $G_1 = G$ and let $G_2 = (V_2, E_2)$ be as follows:

- $V_2 = \{v_1, v_2, v_3\}$, and
- E_2 consists of exactly the 3 (undirected) edges $[v_1, v_2]$, $[v_2, v_3]$ and $[v_1, v_3]$.

Task: Prove the "$\Leftarrow$" direction in the proof of correctness of the reduction, i.e. prove the following statement: if (G_1, G_2) is a positive instance of **HOM**, then G is a positive instance of **3-COL**.

Note: For any property that you use in your proof, make it perfectly clear why this property holds (e.g., "by the problem reduction", "by the assumption X", "by the definition X", etc.)

(15 points)

2.) (a) First define the concept of a $\mathcal{T}$-interpretation. Then use it to define the following:

i. the $\mathcal{T}$-satisfiability of a formula;

ii. the $\mathcal{T}$-validity of a formula.

Additionally define the completeness of a theory $\mathcal{T}$ and give an example for a complete and an incomplete theory. **(4 points)**

(b) Let $\mathcal{T}_E^{di}$ be a first-order theory containing all axioms of the theory of equality $\mathcal{T}_E$ and the following two axioms:

$$\forall x \forall y \Big(p(x, y) \rightarrow \big(p(x, f(x, y)) \land p(f(x, y), y) \big) \Big) \qquad \text{(p-density)}$$

$$\forall x \forall y \Big(p(x, y) \rightarrow x \neq y \Big) \qquad \text{(p-irreflexivity)}$$

Prove: Let I be a $\mathcal{T}_E^{di}$-interpretation with $I \models p(a, b)$, then it holds that $I \models f(a, b) \neq a \land f(a, b) \neq b \land a \neq b$. **(8 points)**

(c) Apply Ackermann's reduction to the following EUF-formula ψ:

$$p(a, F(b)) \land F(F(c)) \doteq G(G(b)) \rightarrow p(a, c)$$

Hint: Treat uninterpreted predicates correctly. **(3 points)**

3.) Prove that the following correctness assertion is true regarding total correctness. Use the invariant $l * y \leq x < h * y \wedge y > 0$.

Some annotation rules you might need:
$\{F\}v := e \mapsto \{F\}v := e\{\exists v'(F[v/v'] \wedge v = e[v/v'])\}$
if e then $\{F\} \cdots$ else $\{G\} \mapsto \{(e \Rightarrow F) \wedge (\neg e \Rightarrow G)\}$if e then $\{F\} \cdots$ else $\{G\}$
$\{F\}$if e then $\cdots$ else $\mapsto \{F\}$if e then $\{F \wedge e\} \cdots$ else $\{G \wedge \neg e\}$
while e do $\cdots$ od $\mapsto \{\, Inv \,\}$while e do $\{\, Inv \wedge e \wedge t{=}t_0 \,\} \cdots \{\, Inv \wedge (e \Rightarrow 0{\leq}t{<}t_0) \,\}od\{\, Inv \wedge \neg e \,\}$

$\{\, y > 0 \wedge x \geq 0 \,\}$
$l := 0;$
$h := x + 1;$
while $l + 1 \neq h$ do
    $z := (l + h)/2;$
    if $z * y > x$ then
        $h := z$
    else
        $l := z;$
    fi
od
$\{\, l * y \leq x < (l + 1) * y \,\}$

(15 points)

4.) (a) Find a Kripke structure K with initial state s_0 that has the properties $\mathbf{AGEF}p$ and $\mathbf{A}\,(\mathbf{GF}p \Rightarrow \mathbf{GF}q)$ at state s_0, but not $\mathbf{AG}(p \Rightarrow \mathbf{AF}q)$. Justify your choice. **(5 points)**

(b) Given a graph, write a C program such that CBMC can determine whether the given graph is 3-colorable. Augment the given code corresponding to the following subtasks. The 2-dimensional array `graph` encodes the adjacency matrix.

```
#define TRUE  1
#define FALSE 0

#define RED   0
#define GREEN 1
#define BLUE  2

#define N     4 // Number of nodes in the graph

int graph[N][N] = { { 0, 1, 0, 1 }, { 1, 0, 0, 0 }, ... };
int coloring[N];

int nondet_int();
```

i. Write a loop that nondeterministically guesses a coloring for the graph. A coloring assigns to every node of the given graph either the color red, green, or blue.
ii. Write a loop that checks whether the coloring assigns to every node in the graph a color that is different to the colors of its neighbors. Furthermore, ensure that CBMC reports a 3-coloring of the graph in case there exists one.

(6 points)

(c) Show that simulation is a transitive relation: Given any 3 Kripke structures $K_1 = \{S_1, R_1, L_1\}$, $K_2 = \{S_2, R_2, L_2\}$ and $K_3 = \{S_3, R_3, L_3\}$ such that $K_1 \leq K_2$ and $K_2 \leq K_3$, it holds that $K_1 \leq K_3$.

(4 points)

DEFINITIONS

Let $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ be two Kripke structures.

Simulation

A relation $H \subseteq S_1 \times S_2$ is a simulation relation if for each $(s, s') \in H$ holds:

- $L_1(s) = L_2(s')$, and
- for each $(s, t) \in R_1$ there is a $(s', t') \in R_2$ such that $(t, t') \in H$.

M_2 *simulates* M_1 (denoted as $M_1 \leq M_2$), if there is a simulation relation $H \subseteq S_1 \times S_2$ *such that*

- for each initial state $s \in I_1$ there is an initial state $s' \in I_2$ with $(s, s') \in H$.