Logic and Computability VU 2012W

Exercise Session 2 – *Modal Logic, Intuitionistic Logic, Automated Deduction*

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Exercise *Find some further examples of modal formulas with one schematic variable that are valid in $\mathcal{F}$, as above, such that the removal of some (which?) accessibilities leads to invalidity.*

Solution. Consider the following:

1. It is easy to see that $\Diamond(A \wedge A)$, $\Diamond(A \vee \neg A)$ are valid in $\mathcal{F}$, but become invalid if, e.g., $R(w, w)$ and $R(w, u)$ are removed since the operator $\Diamond$ states an existential claim for an accessible world.
2. The formula scheme $\Box A \supset A$ is valid in $\mathcal{F}$, but ceases to be valid if $R(w, w)$ is removed.
3. The formula scheme $\Box A \supset \Diamond A$ is valid in $\mathcal{F}$, but is not valid if $R(w, w)$ and $R(u, w)$ are removed.
4. The formula scheme $A \supset \Box\Diamond A$ is valid in $\mathcal{F}$, but is not valid if $R(w, u)$ is removed.
5. The formula scheme $\Box A \supset \Box\Box A$ is valid in $\mathcal{F}$, but is not valid if $R(w, w)$ is removed.

$\triangle$

Exercise *Show that the intersection of two logics is also a logic. What about the union of logics?*

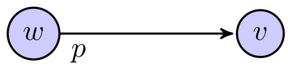
Solution. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be logics, π be an arbitrary substitution and for every formula F, let $F[\pi]$ denote the formula resulting from F by successively applying the substitution π. Let $F \in \mathcal{L}_1 \cap \mathcal{L}_2$. First, consider $F[\pi]$. Since $F \in \mathcal{L}_1$ and $F \in \mathcal{L}_2$, we have by hypotheses that $F[\pi] \in \mathcal{L}_1$ and $F[\pi] \in \mathcal{L}_2$, hence (by basic set theory) we immediately infer $F[\pi] \in \mathcal{L}_1 \cap \mathcal{L}_2$. Now let $A \in \mathcal{L}_1 \cap \mathcal{L}_2$ and $A \supset B \in \mathcal{L}_1 \cap \mathcal{L}_2$. Since again

$A, A \supset B \in \mathcal{L}_1$ and $A, A \supset B \in \mathcal{L}_2$, we obtain by hypothesis that $B \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$ and hence, $B \in \mathcal{L}_1 \cap \mathcal{L}_2$. Summing up, we showed that $\mathcal{L}_1 \cap \mathcal{L}_2$ is both closed under substitution and under MP, hence, $\mathcal{L}_1 \cap \mathcal{L}_2$ constitutes a logic by the very definition.

The union of logics is in the general case not a logic again. For that, consider $\mathcal{L}_1 = \{\bot\}$ and $\mathcal{L}_2 = \{\bot \supset \top\}$. Obviously, $\mathcal{L}_1$ and $\mathcal{L}_2$ constitute both a logic in their own sense. But $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 = \{\bot \supset \top, \bot\}$ is not a logic, since $\bot \supset \top \in \mathcal{L}$ and $\bot \in \mathcal{L}$, but $\top \notin \mathcal{L}$, i.e., $\mathcal{L}$ is not closed under MP. $\triangle$

Exercise *Find a counter-example to $F \supset \Box F$.*

Solution. Let $F = p$ for some $p \in PV$ and $\mathcal{M} = \langle W, R, V \rangle$ a Kripke interpretation given as follows (those and only those propositional variables are mentioned at each world which are true in that world):



Obviously we have that $\mathcal{M}, w \models p$, but clearly we do not have $\mathcal{M}, w \models \Box p$ and, hence, $\mathcal{M}, w \not\models p \supset \Box p$ and $\not\models p \supset \Box p$ as required. $\triangle$

Exercise *Prove formally that $\Box A \supset A$ characterizes reflexivity. Prove that $\Box A \supset \Box\Box A$ characterizes transitivity.*

Solution. Given a frame $\mathcal{F} = \langle W, R \rangle$ and some world $w \in W$, let $R|w := \{v \mid (w, v) \in R\}$. First, we consider reflexivity:

($\Leftarrow$): Suppose that $\mathcal{F} = \langle W, R \rangle$ is reflexive. Then, for all $w \in W$ it holds that wRw. Now let $w \in W$ be an arbitrary world and $\mathcal{M}$ an arbitrary interpretation based on $\mathcal{F}$. Suppose we have that $\mathcal{M}, w \models \Box A$. By semantics, we obtain that $\mathcal{M}, u \models A$ for every world $u \in W$ such that wRu. Since $\mathcal{F}$ is reflexive we also have $\mathcal{M}, w \models A$, i.e., if $\mathcal{F} \models \Box A$ then $\mathcal{F} \models A$ and $\mathcal{F} \models \Box A \supset A$ as desired.

($\Rightarrow$): We proceed indirectly. Suppose that $\mathcal{F} = \langle W, R \rangle$ is not reflexive. Then, there must be some world $w \in W$ such that $(w, w) \notin R$. Define an assignment $V(p, v) = \mathbf{1}$ iff $v \in R|w$. Let $\mathcal{M} = \langle W, R, V \rangle$. Obviously we have by def. of $\mathcal{M}$ that $\mathcal{M}, w \models \Box p$ but $\mathcal{M}, w \not\models p$. Hence, $\mathcal{F} \not\models \Box p \supset p$ as required.

Now consider transitivity:

($\Leftarrow$): Suppose that $\mathcal{F}$ is transitive. Then, for every $s, t, u \in W$ it holds that if sRt and tRu we have sRu. Now let $\mathcal{M}$ be a model based on $\mathcal{F}$ and let $w \in W$ be an arbitrary world. Suppose that $\mathcal{M}, w \models \Box A$. Then, $\mathcal{M}, v \models A$ for every world $v \in W$ such that wRv. If $\mathcal{M}, w \not\models \Box\Box A$, then there must be some world $t \in W$ such that wRt and $\mathcal{M}, t \not\models \Box A$. Thus, there must be some world $u \in W$ such that tRu but $\mathcal{M}, u \not\models A$. But by transitivity we also obtain that wRu and, hence, $\mathcal{M}, u \models A$ has to hold by hypothesis. Hence, if $\mathcal{M}, w \models \Box A$, then $\mathcal{M}, w \models \Box\Box A$ and we obtain $\mathcal{F} \models \Box A \supset \Box\Box A$ as desired.

($\Rightarrow$): Suppose that $\mathcal{F} \models \Box A \supset \Box\Box A$ holds. We proceed indirectly. Assume that $\mathcal{F}$ is not transitive. Then there must be some $s, t, u \in W$ such that sRt and tRu but not sRu. We define an assignment V such that $\langle W, R, V\rangle \not\models \Box A \supset \Box\Box A$ for some A. For some $p \in PV$ define $V(p, v) = \mathbf{1}$ iff $v \in R|s$. By construction, we have that $\langle W, R, V\rangle, s \models \Box p$, but since $R|t$ is not empty, we also have $\langle W, R, V\rangle, s \not\models \Box\Box p$ since $(s, u) \notin R$. Hence, $\mathcal{F} \not\models \Box p \supset \Box\Box p$ as required. $\triangle$

Exercise

1. *Find (at least one) appropriate internet resource for 'bisimulation' as well as for 'bounded morphism' (also called 'p-morphism').*
2. *Summarize the central definition and fact(s) precisely.*
3. *Give non-trivial examples of bisimilar models.*
4. *Apply a bounded morphism to show that asymmetry is not characterizable.*

Solution. Ad (1): my sources are

(i) `http://en.wikipedia.org/wiki/Kripke_semantics` and

(ii) `http://www.mathematik.tu-darmstadt.de/~otto/papers/mlhb.pdf`, paper by Goranko et. al.

We begin with (2) and define the central terms:

In the following let $\mathcal{F}_1 = \langle W, R\rangle$ and $\mathcal{F}_2 = \langle W', R'\rangle$ be Kripke frames and let $\mathcal{M}_1 = \langle W, R, V\rangle$ be a Kripke model based on $\mathcal{F}_1$ and $\mathcal{M}_2 = \langle W', R', V'\rangle$ a Kripke model based on $\mathcal{F}_2$. A *bisimulation* between $\mathcal{F}_1$ and $\mathcal{F}_2$ is a non-empty relation $\rho \subseteq W \times W'$ such that the following conditions hold for every $w\rho w'$:

(i) if wRu for some $u \in W$, then there is some $u' \in W'$ such that $w'R'u'$ and $u\rho u'$.

(ii) if $w'R'u'$ for some $u' \in W'$, then there is some $u \in W$ such that wRu and $u\rho u'$.

A bisimulation ρ between to Kripke interpretations $\mathcal{M}_1$ and $\mathcal{M}_2$ additionally has to satisfy the following condition for every propositional atom p and any $w\rho w'$:

(i) $\mathcal{M}_1, w \models p$ iff $\mathcal{M}_2, w' \models p$.

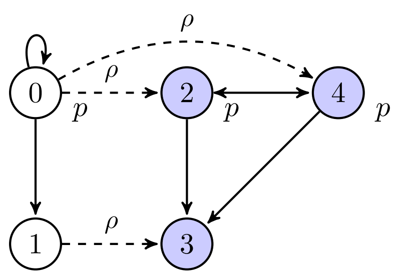
Two Kripke frames (interpretations) $\mathcal{F}_1, \mathcal{F}_2$ ($\mathcal{M}_1, \mathcal{M}_2$) are called *bisimilar* if there exists a bisimulation ρ between $\mathcal{F}_1$ and $\mathcal{F}_2$ ($\mathcal{M}_1$ and $\mathcal{M}_2$).

Bounded morphisms or *p-morphisms* are special cases of bisimulations. A function $\rho : W \mapsto W'$ is a *bounded morphism* if its graph $\{(w, v) \in W \times W' \mid v = \rho(w)\}$ is a bisimulation. Bounded morphisms between frames are similarly defined. Hence, we can conclude the following defining conditions for a bounded morphism $\rho : W \mapsto W'$ between $\mathcal{M}_1$ and $\mathcal{M}_2$:

1. For every atom p, $\mathcal{M}_1, w \models p$ iff $\mathcal{M}_2, \rho(w) \models p$.
2. For any $w \in W$, if wRu for some u, then $\rho(w)R'\rho(u)$.
3. For any $w \in W$, if $\rho(w)R'u'$ for some $u' \in W'$, then there is some $u \in W$ such that $\rho(u) = u'$ and wRu.

If a bounded morphism ρ between $\mathcal{M}_1$ and $\mathcal{M}_2$ is onto, then we say that $\mathcal{M}_2$ is a *bounded morphic image* of $\mathcal{M}_1$ (for frames likewise).

The following two models are bisimilar (the bisimulation ρ is represented by dashed arrows):



The central result, which we will also use for showing that asymmetry is not characterizable, is the following

Theorem. *Let $\mathcal{F}_1 = \langle W, R\rangle, F_2 = \langle W', R'\rangle$ be Kripke frames and $\rho : W \mapsto W'$ a p-morphism. If ρ is one-to-one, then $\mathcal{F}_2 \models A$ implies $\mathcal{F}_1 \models A$. If ρ is onto, then $\mathcal{F}_1 \models A$ implies $\mathcal{F}_2 \models A$.*

Now suppose that there is a formula A such that $\mathcal{F} \models A$ iff $\mathcal{F}$ is asymmetric. Let $\mathcal{F}_1 = \langle W, R\rangle$ and $\mathcal{F}_2 = \langle W', R'\rangle$ such that

$$W = \{a, b\}, \qquad R = \{(a, a), (b, a)\},$$
$$W' = \{c\}, \qquad R' = \{(c, c)\}.$$

Clearly, F_1 is asymmetric, since $(a, b) \notin R$, but F_2 is obviously symmetric. Let $\rho : W \mapsto W'$ a function such that $\rho(w) = c$ for every $w \in W$. Clearly, ρ constitutes a p-morphism and is onto. Since $\mathcal{F}_1$ is asymmetric, it must hold that $\mathcal{F}_1 \models A$. By the theorem above, it must also hold that $\mathcal{F}_2 \models A$. But $\mathcal{F}_2$ is not asymmetric and, hence, there can be no such formula A which characterizes asymmetry. $\triangle$

Exercise *Show analogously the validity of formulas 3 and 7.*

Solution. We begin with formula 3: $\gamma : (A \wedge B) \supset (B \wedge A)$ means procedure γ transforms every proof ρ of $A \wedge B$ into a proof π of $B \wedge A$. Let ρ be the pair $\langle \delta, \eta \rangle$ where $\delta : A$ and $\eta : B$. The procedure γ is constructed in the following way: extract δ from ρ and η from ρ and construct the proof $\pi : \langle \eta, \delta \rangle$. Then, clearly, we have that $\pi : B \wedge A$ and γ is a procedure which transforms $\rho : A \wedge B$ into $\pi : B \wedge A$.

Now consider formula 7: $\gamma : \neg(A \vee B) \supset (\neg A \wedge \neg B)$ means procedure γ transforms a proof ρ of $(A \vee B) \supset \bot$ into a proof π of $(A \supset \bot) \wedge (B \supset \bot)$. Clearly, ρ is a refutation of $A \vee B$, i.e., ρ can construct from any proof of $A \vee B$ a proof of $\bot$. The proof π is constructed from ρ in the following way: let $\eta_1 = \langle l, \nu_1 \rangle$ and $\eta_2 = \langle r, \nu_2 \rangle$ where $\nu_1 : A$ and $\nu_2 : B$. Clearly, $\eta_1 : A \vee B$ and $\eta_2 : A \vee B$. Now we let $\pi = \langle \rho(\eta_1), \rho(\eta_2) \rangle$, i.e., we apply the refutation ρ to both η_1 and η_2. Then, $\rho(\eta_1) : A \supset \bot$ and $\rho(\eta_2) : B \supset \bot$ and π constitutes a proof of $(A \supset \bot) \wedge (B \supset \bot)$. $\triangle$

Exercise *Show the validity of the following formula in* LK: $sk(\forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)) \to \forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)$

Solution. We first skolemize the formula $\forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)$ step-by-step:

$$
\begin{aligned}
& sk(\forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)) \\
& \rightsquigarrow \forall x \forall y \exists z \forall u F(x, y, z, u, f(x, y, z, u)) \\
& \rightsquigarrow \forall x \forall y \forall u F(x, y, g(x, y), u, f(x, y, g(x, y), u)).
\end{aligned}
$$

Now consider the following proof in LK:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash F(x,y,g(x,y),u,f(x,y,g(x,y),u))}}{F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \exists v F(x,y,g(x,y),u,v)}\,(\exists,r)}{\forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \exists v F(x,y,g(x,y),u,v)}\,(\forall,l)}{\forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \forall u \exists v F(x,y,g(x,y),u,v)}\,(\forall,r)}{\forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \exists z \forall u \exists v F(x,y,z,u,v)}\,(\exists,r)}{\forall y \forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \exists z \forall u \exists v F(x,y,z,u,v)}\,(\forall,l)}{\forall x \forall y \forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \exists z \forall u \exists v F(x,y,z,u,v)}\,(\forall,l)}{\forall x \forall y \forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \forall y \exists z \forall u \exists v F(x,y,z,u,v)}\,(\forall,r)}{\forall x \forall y \forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \vdash \forall x \forall y \exists z \forall u \exists v F(x,y,z,u,v)}\,(\forall,r)}{\vdash \forall x \forall y \forall u F(x,y,g(x,y),u,f(x,y,g(x,y),u)) \to \forall x \forall y \exists z \forall u \exists v F(x,y,z,u,v)}\,(\to,r)
$$

$\triangle$

Exercise *Prove that if* $\forall x_1 \cdots \forall x_n F$ *is satisfiable, then* $\forall x_1 \cdots \forall x_n \delta(F)$ *is satisfiable.*

Solution. First notice that by the lemma about the definitions, it suffices to show that if $\bigwedge_{G \in \Sigma(F)} E_G$ is satisfiable, then $\bigwedge_{G \in \Sigma(F)} D_G$ is satisfiable too. We prove the statement by induction on the logical complexity of F. In the following let $\mathcal{X} = \{x_1, \dots, x_k\}$ be the set of free variables from G. For some formula F, we shall write $\forall \mathcal{X} F$ instead of $\forall x_1 \cdots \forall x_k F$. For an arbitrary formula F, let $\Sigma(F)$ be the set of all subformulas of F and let $\Sigma^E_\wedge(F) := \bigwedge_{G \in \Sigma(F)} E_G$ and $\Sigma^D_\wedge(F) := \bigwedge_{G \in \Sigma(F)} D_G$, where E_G and D_G are constructed like in the lecture slides.

For the base case, assume that F is an atomic formula and assume that $\Sigma^E_\wedge(F) = E_F$ is satisfiable. Then E_F is of the form $\forall \mathcal{X}(p_F(x_1, \dots, x_k) \leftrightarrow F)$ and $\Sigma^D_\wedge(F) = D_F$ is of the form $\forall \mathcal{X}(\neg p_F(\mathcal{X}) \vee F) \wedge \forall \mathcal{X}(p_F(\mathcal{X}) \vee \neg F)$. By using the equivalences (which are easily proved)

$$\begin{aligned} A \leftrightarrow B \equiv (A \to B) \wedge (B \to A) &\equiv (A \vee \neg B) \wedge (\neg A \wedge B), \\ \forall \mathcal{X} A \wedge \forall \mathcal{X} B &\equiv \forall \mathcal{X}(A \wedge B), \end{aligned}$$

we immediately obtain by equivalence replacement theorem of classical logic, that $E_F \equiv D_F$. Hence, since E_F is satisfiable, D_F must be satisfiable too.

Now assume that $F = F_1 \star F_2$ ($\star \in \{\wedge, \vee, \to\}$) has complexity $n+1$, where F_1, F_2 are formulas of complexity less or equal to n and assume that for $\varphi \in \{F_1, F_2\}$ if $\Sigma^E_\wedge(\varphi)$ is satisfiable, then $\Sigma^D_\wedge(\varphi)$ is satisfiable and that $\Sigma^E_\wedge(F)$ is satisfiable. Let $\{y_1, \dots, y_l\} \subseteq \mathcal{X}$ be the free variables of F_1 and $\{z_1, \dots, z_m\} \subseteq \mathcal{X}$ those of F_2. Then, E_F is of the form

$$\forall \mathcal{X}(p_F(\mathcal{X}) \leftrightarrow (p_{F_1}(y_1, \dots, y_l) \star p_{F_2}(z_1, \dots, z_m))). \tag{1}$$

Consider the following equivalences for $\wedge$:

$$\begin{aligned} A \leftrightarrow (B \wedge C) &\equiv (A \to (B \wedge C)) \wedge ((B \wedge C) \to A) \equiv \\ & (\neg A \vee (B \wedge C)) \wedge (\neg(B \wedge C) \vee A) \equiv \\ & (\neg A \vee B) \wedge (\neg A \vee C) \wedge (A \vee \neg B \vee \neg C). \end{aligned}$$

If $F = F_1 \wedge F_2$, then D_F must be of the form

$$\forall \mathcal{X}(\neg p_F(\mathcal{X}) \vee p_{F_1}(\mathcal{X})) \wedge \forall \mathcal{X}(\neg p_F(\mathcal{X}) \vee p_{F_2}(\mathcal{X})) \wedge \forall \mathcal{X}(p_F(\mathcal{X}) \vee \neg p_{F_1}(\mathcal{X}) \vee \neg p_{F_2}(\mathcal{X})). \tag{2}$$

It is obvious that (1) is equivalent to (2) by equivalence replacement theorem. Furthermore, the equivalence of (1) and (2), the construction of D_F, and the hypotheses give us that $\Sigma^D_\wedge(F)$ is satisfiable.[1] The cases for $\vee, \to$ and $\neg$ are treated similarly (for negation, of course $F = \neg F_1$ and (1) is of the form as given in the slides). We just give the equivalences:

$$\begin{aligned} A \leftrightarrow (B \vee C) &\equiv (A \to (B \vee C)) \wedge ((B \vee C) \to A) \equiv \\ & (\neg A \vee B \vee C) \wedge (\neg(B \vee C) \vee A) \equiv \\ & (\neg A \vee B \vee C) \wedge (A \vee \neg B) \wedge (A \vee \neg C). \end{aligned}$$

[1] Of course, this argument does not rigorously establish the connection between $\Sigma^D_\wedge(F)$ and $\Sigma^E_\wedge(F)$. However, it is easy to see that the "transformations" which occur between D_F and E_F preserve satisfiability because of (i) equivalent transformations in the case of propositional connectives and (ii) Skolemization in the case of quantifiers. Hence, we only justify the satisfiability equivalence of these transformations.

$$A \leftrightarrow (B \to C) \equiv (A \to (B \to C)) \land ((B \to C) \to A) \equiv$$
$$(\neg A \lor \neg B \lor C) \land (\neg(\neg B \lor C) \lor A) \equiv$$
$$(\neg A \lor \neg B \lor C) \land (A \lor B) \land (A \lor \neg C).$$

$$A \leftrightarrow \neg B \equiv (A \to \neg B) \land (\neg B \to A) \equiv$$
$$(\neg A \lor \neg B) \land (B \lor A).$$

Consider the case where $F = \mathsf{Q}uF_1$ (for $\mathsf{Q} \in \{\forall, \exists\}$). Then, D_F is of the form

$$\forall \mathcal{X}(p_F(\mathcal{X}) \leftrightarrow \mathsf{Q}vp_{F_1}(\mathcal{X}, v)).$$

Consider the equivalences for the universal quantifier (suppose that u does not occur free in A):

$$\forall \mathcal{X}(A(\mathcal{X}) \leftrightarrow \forall uB(\mathcal{X}, u)) \equiv \forall \mathcal{X}((A(\mathcal{X}) \to \forall uB(\mathcal{X}, u)) \land (\forall uB(\mathcal{X}, u) \to A(\mathcal{X}))) \equiv$$
$$\forall \mathcal{X}((\neg A(\mathcal{X}) \lor \forall uB(\mathcal{X}, u)) \land (\exists u \neg B(\mathcal{X}, u) \lor A(\mathcal{X}))) \equiv$$
$$\forall \mathcal{X} \forall u(\neg A(\mathcal{X}) \lor B(\mathcal{X}, u)) \land \forall \mathcal{X}(\exists u \neg B(\mathcal{X}, u) \lor A(\mathcal{X})).$$

Skolemization results in the satisfiability equivalent formula $\forall \mathcal{X} \forall u(\neg A(\mathcal{X}) \lor B(\mathcal{X}, u)) \land \forall \mathcal{X}(\neg B(\mathcal{X}, f(\mathcal{X})) \lor A(\mathcal{X}))$.

At last, consider the case of the existential quantifier:

$$\forall \mathcal{X}(A(\mathcal{X}) \leftrightarrow \exists uB(\mathcal{X}, u)) \equiv \forall \mathcal{X}((A(\mathcal{X}) \to \exists uB(\mathcal{X}, u)) \land (\exists uB(\mathcal{X}, u) \to A(\mathcal{X}))) \equiv$$
$$\forall \mathcal{X}((\neg A(\mathcal{X}) \lor \exists uB(\mathcal{X}, u)) \land (\forall u \neg B(\mathcal{X}, u) \lor A(\mathcal{X}))) \equiv$$
$$\forall \mathcal{X}(\neg A(\mathcal{X}) \lor \exists uB(\mathcal{X}, u)) \land \forall \mathcal{X} \forall u(\neg B(\mathcal{X}, u) \lor A(\mathcal{X})).$$

Skolemization results in the satisfiability equivalent formula $\forall \mathcal{X}(\neg A(\mathcal{X}) \lor B(\mathcal{X}, f(\mathcal{X}))) \land \forall \mathcal{X} \forall u(\neg B(\mathcal{X}, u) \lor A(\mathcal{X}))$. △

Exercise *Compute $\delta(F)'$ for $F = \exists y(p(x, g(x, y)) \to \exists z \neg(p(g(x, y), z) \lor p(y, x)))$.*

Solution. Let

$$F_1 = p(x, g(x, y)) \to \exists z \neg(p(g(x, y), z) \lor p(y, x)),$$
$$F_2 = \exists z \neg(p(g(x, y), z) \lor p(y, x)),$$
$$F_3 = \neg(p(g(x, y), z) \lor p(y, x)),$$
$$F_4 = p(g(x, y), z) \lor p(y, x).$$

Now consider the set $\Sigma(F)$ consisting of all non-atomic subformulas. We have that

$$\Sigma(F) = \{F\} \cup \{F_i \,|\, 1 \le i \le 5\}.$$

For each $G \in \Sigma(F)$, we shall introduce some new atom $p_G(x_1, \ldots, x_k)$ where $x_1, \ldots, x_k$ are the corresponding free variables of G. We introduce the following definitions:

$$\begin{aligned}
G_1 &:= \forall x \forall y \forall z (p_{F_4}(x, y, z) \leftrightarrow (p(g(x, y), z) \vee p(y, x))), \\
G_2 &:= \forall x \forall y \forall z (p_{F_3}(x, y, z) \leftrightarrow \neg p_{F_4}(x, y, z)), \\
G_3 &:= \forall x \forall y (p_{F_2}(x, y) \leftrightarrow \exists u \, p_{F_3}(x, y, u)), \\
G_4 &:= \forall x \forall y (p_{F_1}(x, y) \leftrightarrow (p(x, g(x, y)) \rightarrow p_{F_2}(x, y))), \\
G_5 &:= \forall x (p_F(x) \leftrightarrow \exists u \, p_{F_1}(x, u)).
\end{aligned}$$

Now we proceed translating each defining formula to CNF:

$$\begin{aligned}
D_{G_1} \rightsquigarrow\ & \forall x \forall y \forall z (\neg p_{F_4}(x, y, z) \vee p(g(x, y), z) \vee p(y, x)) \wedge \\
& \forall x \forall y \forall z (p_{F_4}(x, y, z) \vee \neg p(g(x, y), z)) \wedge \forall x \forall y \forall z (p_{F_4}(x, y, z) \vee \neg p(y, x)), \\
D_{G_2} \rightsquigarrow\ & \forall x \forall y \forall z (\neg p_{F_3}(x, y, z) \vee \neg p_{F_4}(x, y, z)) \wedge \\
& \forall x \forall y \forall z (p_{F_3}(x, y, z) \vee p_{F_4}(x, y, z)), \\
D_{G_3} \rightsquigarrow\ & \forall x \forall y (\neg p_{F_2}(x, y) \vee p_{F_3}(x, y, f(x, y))) \wedge \\
& \forall x \forall y \forall u (p_{F_2}(x, y) \vee \neg p_{F_3}(x, y, u)), \\
D_{G_4} \rightsquigarrow\ & \forall x \forall y (\neg p_{F_1}(x, y) \vee \neg p(x, g(x, y)) \vee p_{F_2}(x, y)) \wedge \\
& \forall x \forall y (p_{F_1}(x, y) \vee p(x, g(x, y))) \wedge \forall x \forall y (p_{F_1}(x, y) \vee \neg p_{F_2}(x, y)), \\
D_{G_5} \rightsquigarrow\ & \forall x (\neg p_F(x) \vee p_{F_1}(x, h(x))) \wedge \forall x \forall u (p_F(x) \vee \neg p_{F_1}(x, u)).
\end{aligned}$$

Now we can construct our set of clauses $\delta(F)'$:

$$\begin{aligned}
\delta(F)' = \{ & \neg p_{F_4}(x, y, z) \vee p(g(x, y), z) \vee p(y, x), p_{F_4}(x, y, z) \vee \neg p(g(x, y), z), \\
& p_{F_4}(x, y, z) \vee \neg p(y, x), \neg p_{F_3}(x, y, z) \vee \neg p_{F_4}(x, y, z), p_{F_3}(x, y, z) \vee p_{F_4}(x, y, z), \\
& \neg p_{F_2}(x, y) \vee p_{F_3}(x, y, f(x, y)), p_{F_2}(x, y) \vee \neg p_{F_3}(x, y, u), p_{F_1}(x, y) \vee \neg p_{F_2}(x, y), \\
& \neg p_{F_1}(x, y) \vee \neg p(x, g(x, y)) \vee p_{F_2}(x, y), p_{F_1}(x, y) \vee p(x, g(x, y)) \\
& \neg p_F(x) \vee p_{F_1}(x, h(x)), p_F(x) \vee \neg p_{F_1}(x, u) \}.
\end{aligned}$$

$\triangle$

Exercise R1 *Find all Robinson-resolvents of* $C = p(x, f(x)) \vee p(a, y)$ *and* $D = \neg p(x, y) \vee \neg p(a, f(x)) \vee \neg p(f(x), f(y))$. *Specify all used renamings, mgus and (implicit) factors.*

Solution. Since every clause is a (trivial) factor of itself, we first consider Robinson-resolvents which are binary resolvents. Consider the following:

- Let $C' = p(z, f(z)) \vee p(a, u)$ be a variant of C (apply renaming $\nu = \{x \mapsto z, y \mapsto u\}$). We can resolve upon the second literal of C' and the first of D using the mgu $\sigma = \{x \mapsto a, y \mapsto u\}$ obtaining the Robinson-resolvent $p(z, f(z)) \vee \neg p(a, f(a)) \vee \neg p(f(a), f(u))$.

- Let $C' = p(z, f(z)) \vee p(a, u)$ be a variant of C (apply renaming $\nu = \{x \mapsto z, y \mapsto u\}$). We can resolve upon the first literal of C' and the second of D using the mgu $\sigma = \{z \mapsto a, x \mapsto a\}$ obtaining the Robinson-resolvent $p(a, u) \vee \neg p(a, y) \vee \neg p(f(a), f(y))$.
- Let $C' = p(z, f(z)) \vee p(a, u)$ be a variant of C (apply renaming $\nu = \{x \mapsto z, y \mapsto u\}$). We can resolve upon the second literal of C' and the second of D using the mgu $\sigma = \{u \mapsto f(x)\}$ obtaining the Robinson-resolvent $p(x, f(x)) \vee \neg p(x, y) \vee \neg p(f(x), f(y))$.
- Let $C' = p(z, f(z)) \vee p(a, u)$ be a variant of C (apply renaming $\nu = \{x \mapsto z, y \mapsto u\}$). We can resolve upon the first literal of C' and the third of D using the mgu $\sigma = \{z \mapsto f(x), y \mapsto f(x)\}$ obtaining the Robinson-resolvent $p(a, u) \vee \neg p(x, f(x)) \vee \neg p(a, f(x))$.
- Let $C' = p(z, f(z)) \vee p(a, u)$ be a variant of C (apply renaming $\nu = \{x \mapsto z, y \mapsto u\}$). We can resolve upon the first literal of C' and the first of D using the mgu $\sigma = \{x \mapsto z, y \mapsto f(z)\}$ obtaining the Robinson-resolvent $p(a, u) \vee \neg p(a, f(z)) \vee \neg p(f(z), f(f(z)))$.

Now we consider Robinson-resolvents which emerge from non-trivial factors, i.e., these resolvents emerge from clauses C', D' such that C' is a non-trivial factor of C' and D' is a non-trivial factor of D, respectively:

- Let C' be a factor of C by applying the mgu $\nu = \{x \mapsto a, y \mapsto f(a)\}$, i.e., $C' = p(a, f(a))$. C' and D are variable-disjoint and we may resolve upon the first literal of C' and the first of D' using the substitution $\sigma = \{x \mapsto a, y \mapsto f(a)\}$ obtaining the Robinson-resolvent $\neg p(a, f(a)) \vee \neg p(f(a), f(f(a)))$.
- Let C' be a factor of C by applying the mgu $\nu = \{x \mapsto a, y \mapsto f(a)\}$, i.e., $C' = p(a, f(a))$. C' and D are variable-disjoint and we may resolve upon the first literal of C' and the second of D' using the substitution $\sigma = \{x \mapsto a\}$ obtaining the Robinson-resolvent $\neg p(a, y) \vee \neg p(f(a), f(y))$.
- Let D' be a factor of D by applying the mgu $\nu = \{x \mapsto a, y \mapsto f(a)\}$, i.e., $D' = \neg p(a, f(a)) \vee \neg p(f(a), f(f(a)))$. C and D' are variable-disjoint and we may resolve upon the first literal of C and the first of D' by applying the mgu $\sigma = \{x \mapsto a\}$ obtaining the Robinson-resolvent $p(a, y) \vee \neg p(f(a), f(f(a)))$.
- Let D' be a factor of D by applying the mgu $\nu = \{x \mapsto a, y \mapsto f(a)\}$, i.e., $D' = \neg p(a, f(a)) \vee \neg p(f(a), f(f(a)))$. C and D' are variable-disjoint and we may resolve upon the second literal of C and the first of D' by applying the mgu $\sigma = \{y \mapsto f(a)\}$ obtaining the Robinson-resolvent $p(x, f(x)) \vee \neg p(f(a), f(f(a)))$.
- Let D' be a factor of D by applying the mgu $\nu = \{x \mapsto a, y \mapsto f(a)\}$, i.e., $D' = \neg p(a, f(a)) \vee \neg p(f(a), f(f(a)))$. C and D' are variable-disjoint and we may

resolve upon the first literal of C and the second of D' by applying the mgu $\sigma = \{x \mapsto f(a)\}$ obtaining the Robinson-resolvent $p(a, y) \vee \neg p(a, f(a))$.

- Let C' be a factor of C by applying the mgu $\nu = \{x \mapsto a, y \mapsto f(a)\}$, and D' a factor of D by applying the substitution $\eta = \{x \mapsto a, y \mapsto f(a)\}$, i.e., $C' = p(a, f(a))$ and $D' = \neg p(a, f(a)) \vee \neg p(f(a), f(f(a)))$. C' and D' are variable-disjoint and we do not have to apply any substitution to unify C' and D' upon the first literals of each, obtaining the Robinson-resolvent $\neg p(f(a), f(f(a)))$.

$\triangle$

Exercise R2 *Prove that subsumed Robinson-resolvents can be discarded without sacrificing refutational completeness.*

Solution. Let $\mathcal{S}$ be an unsatisfiable set of clauses. Consider the resolution operator $\mathcal{R}(\mathcal{S})$ ($\mathcal{R}^n(\mathcal{S})$, respectively) defined in the lecture and consider the (finite) semantic tree $\hat{\mathcal{T}}(\mathcal{S})$ defined for some unsatisfiable set of clauses $\mathcal{S}$. The central observation is that, if some clause C subsumes another clause D, then C fails at every node where D fails. Suppose D fails at node n, i.e., there is a path $\ell_1, \ldots, \ell_k$, where the ℓ_i ($i = 1, \ldots, k$) are literals, such that for some Herbrand instance $D\sigma$, the dual of every literal in $D\sigma$ occurs in $\ell_1, \ldots, \ell_k$. Since C subsumes D, there exists a substitution ν such that $C\nu$ is a subclause of D. From that we immediately obtain that $C\nu\sigma$ is a Herbrand instance which witnesses that C fails at node n. Hence, $\hat{\mathcal{T}}(\mathcal{R}(\mathcal{S}))$ is then also strictly smaller than $\hat{\mathcal{T}}(\mathcal{S})$ when we discard each subsumed clause from $\mathcal{R}(\mathcal{S})$. Thus, we must have that for some $n \geq 0$ that $\hat{\mathcal{T}}(\mathcal{R}^n(\mathcal{S}))$ consists only of the root of $\mathcal{T}(\mathcal{S})$ whereupon we can conclude that $\Box \in \mathcal{R}^n(\mathcal{S})$, since only the empty clause fails at the root of some semantic tree. $\triangle$

Exercise R3 *Describe another resolution refinement (other than ordered resolution) in detail.*

Solution. We shall briefly present *hyperresolution*. We follow the outline given in the book of A. Leitsch: *The Resolution Calculus* and his paper `http://www.logic.at/people/leitsch/resolv.pdf`. Basically each resolution refinement tries to restrict the resolution deduction. Hyperresolution itself is based on the consideration of the *polarity* of clauses, i.e., the polarity of the literals occurring in it. Hyperresolution is now a refinement, where only positive clauses (i.e., clauses of the form $\vdash A_1, \ldots, A_n$) are derivable. The aim is to reduce the search space for further resolvents. Thereby, single-step resolution is replaced by many-step resolution (also called *macro inference*) and many resolution steps are combined to one. We quote the definition of a *hyperresolvent* given by A. Leitsch ($\mathcal{R}es(\mathcal{C})$ denotes the set of resolvents of C and D):

Definition. *Let C be a nonpositive clause and $D_1, \ldots, D_n$ be positive clauses. Then $S : (C; D_1, \ldots, D_n)$ is called a* clash sequence. *Let $C_0 = C$ and $C_{i+1} \in \mathcal{R}es(\{C_i, D_{i+1}\})$ for $i = 1, \ldots, n-1$. If C_n is defined and positive then it is called a* hyperresolvent *of S. The corresponding set of hyperresolvents from a set of clauses $\mathcal{C}$ is denoted by $\varrho(\mathcal{C})$.*

The resolution operator for hyperresolution is denoted by $\mathcal{R}_H$.

Hence, several ordinary steps of resolution deductions *may* result in a hyperresolvent. The completeness of hyperresolution was already shown by J.A. Robinson in his landmark paper from 1965. Hyperresolution can also be combinded with other resolution refinements (e.g. Atom Ordering and Lock Resolution). We give an example of a hyperresolvent given by Prof. Leitsch in his paper. Let $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ where

$$\begin{aligned}
C_1 &= \vdash p(a,b) \\
C_2 &= \vdash p(b,a) \\
C_3 &= p(x,y), p(y,z) \vdash p(x,z) \\
C_4 &= p(a,a) \vdash
\end{aligned}$$

Consider the following resolution refutation:

$$\dfrac{\dfrac{\vdash p(a,b) \qquad \dfrac{\vdash p(b,a) \qquad p(x,y), p(y,z) \vdash p(x,z)}{p(x,b) \vdash p(x,a)}}{\vdash p(a,a)} \qquad p(a,a) \vdash}{\vdash}$$

We have that $S = (C_3; C_1, C_2)$ is a clash-sequence (note that C_1 and C_2 are positive). Furthermore, $\vdash p(a,a)$ constitutes a hyperresolvent of S. $\triangle$