1) Prove that if Σ is correct and $\overline{\mathbf{P}}^*$ is expressible, then Σ is Gödel-incomplete.

Since $\overline{P}^*$ is expressible in Σ there must be some predicate $H \in \mathcal{H}$ such that $n \in \overline{P}^* \iff H(n) \in \mathcal{T}$ holds $\forall n \in \mathcal{N}$. Let h be the Gödel number of H, i.e. $h = \ulcorner H \urcorner$, and let sentence $G \in \mathcal{S}$ be the diagonalization of H, i.e. $G = H(h)$. By definition of $\overline{P}^*$, we have $n \in \overline{P}^* \iff \ulcorner E_n(n) \urcorner \in \overline{P} \quad \forall n \in \mathcal{N}$, and therefore, $h \in \overline{P}^* \iff \ulcorner H(h) \urcorner \in \overline{P} \iff \ulcorner G \urcorner \in \overline{P} \iff \ulcorner G \urcorner \notin P \iff G \notin \mathcal{P}$. Since H expresses $\overline{P}^*$, we also have $h \in \overline{P}^* \iff H(h) \in \mathcal{T} \iff G \in \mathcal{T}$. It follows that G is a Gödel sentence for $\overline{P}$ and $G \notin \mathcal{P} \iff G \in \mathcal{T}$ must hold. Assume $G \notin \mathcal{T}$, then $G \in P$ which is a contradiction since Σ is correct and therefore $P \subseteq T$ must hold. Hence, $G \in \mathcal{T}$ and $G \notin \mathcal{P}$. Assume $G \in \mathcal{R}$, then since Σ is correct and therefore consistent, it follows that $G \notin \mathcal{T}$ which is a contradiction. Hence, $G \notin \mathcal{R}$ and therefore G is undecidable in Σ and Σ is Gödel-incomplete.

2) Prove that if Σ is consistent and $\mathbf{R}^*$ is representable[1], then Σ is Gödel-incomplete.

Since R^* is representable in Σ there must be some predicate $H \in \mathcal{H}$ such that $H(n) \in \mathcal{P} \iff n \in R^*$ holds $\forall n \in \mathcal{N}$. It follows that $H(n) \in \mathcal{P} \iff n \in R^* \iff \ulcorner E_n(n) \urcorner \in R \iff E_n(n) \in \mathcal{R}$ holds $\forall n \in \mathcal{N}$, therefore $H(n) \in \mathcal{P} \iff E_n(n) \in \mathcal{R}$ also holds for $n = \ulcorner H \urcorner = h$. Hence, $H(h) \in \mathcal{P} \iff E_{\ulcorner H \urcorner}(h) \in \mathcal{R} \iff H(h) \in \mathcal{R}$ holds. Assume that $H(h) \in \mathcal{P}$, then also $H(h) \in \mathcal{R}$, which is a contradiction since Σ is consistent and $\mathcal{P} \cap \mathcal{R} = \emptyset$ must hold. Therefore, $H(h) \notin \mathcal{P}$ and $H(h) \notin \mathcal{R}$, therefore $H(h)$ is undecidable in Σ and Σ is Gödel-incomplete.

3) Prove that $\overline{\mathbf{P}}^*$ is not representable in any system Σ.

Assume that there is a system Σ where $\overline{P}^*$ is representable, then by definition there must be some predicate $H \in \mathcal{H}$ such that $H(n) \in \mathcal{P} \iff n \in \overline{P}^*$ holds $\forall n \in \mathcal{N}$. It follows that $H(n) \in \mathcal{P} \iff n \in \overline{P}^* \iff \ulcorner E_n(n) \urcorner \in \overline{P} \iff \ulcorner E_n(n) \urcorner \notin P \iff E_n(n) \notin \mathcal{P}$ holds $\forall n \in \mathcal{N}$, therefore $H(n) \in \mathcal{P} \iff E_n(n) \notin \mathcal{P}$ has to hold also for $n = \ulcorner H \urcorner = h$. However, $H(h) \in \mathcal{P} \iff E_{\ulcorner H \urcorner}(h) \notin \mathcal{P} \iff H(h) \notin \mathcal{P}$ is a contradiction, therefore $\overline{P}^*$ cannot be representable in any system Σ.

[1] $R = \{\ulcorner S \urcorner \mid S \in \mathcal{R}\}$