Solutions of the Exercises

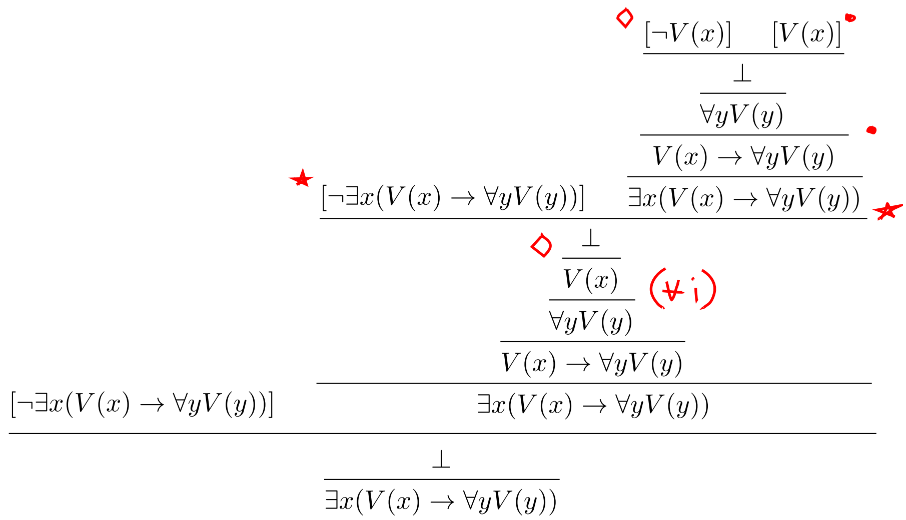
Logic

1. (a) (1 point) Formalize the following sentence in classical logic

   There is a person that if he votes for X then everybody votes for X

   (b) (2 points) exhibit a *natural deduction* derivation for the resulting formula (in case it is valid) or an interpretation that falsifies it (in case it is not valid).

$$\exists x(V(x) \to \forall y V(y))$$



Most common mistakes:

(a) $\exists x V(x) \to \forall y V(y)$

(b) eigenvariable condition for $(\forall i)$

2. (0,5 point) For each of the following statements provide either a proof (if the statement holds) or a counterexample (if the statement does not hold)

   - $\forall x P(x) \to \forall x Q(x) \models \forall x(P(x) \to Q(x))$
   - $\forall x(P(x) \leftrightarrow Q(x))) \models \exists x P(x) \leftrightarrow \exists x Q(x)$
     (where the notation $X \leftrightarrow Y$ abbreviates $(X \to Y) \wedge (Y \to X)$).

   Most students did it right. Sketch of solution:

   - Easy counterexample: $D_{\mathcal{A}} = a, b$, $P^{\mathcal{A}}(a) = Q^{\mathcal{A}}(b) = 1$
   - The statement is true (easy proof)

3. (1 point) Consider the formula

   $$\forall x \exists y A(x, y) \wedge \forall x \neg A(x, x) \wedge \forall xyz[(A(x, y) \wedge A(y, z)) \to A(x, z)]$$

   - Is the formula satisfiable, unsatisfiable or valid?
   - Is the formula satisfiable in interpretations with finite domains?

   Sketch of solution:

   - The formula is satisfiable. For instance: $D_{\mathcal{A}} = \mathcal{N}$, $I^{\mathcal{A}}(A) = \{(x, y) \mid x \in \mathcal{N}, x > y\}$
   - The formula is *not* satisfiable in interpretations with finite domains. One possible argument: given any element $a_i \in D_{\mathcal{A}}$, there must be a_j ($i \neq j$) s.t. $A^{\mathcal{A}}(a_i^{\mathcal{A}}, a_j^{\mathcal{A}})$ holds. The same for a_j and, as $A^{\mathcal{A}}(a_j^{\mathcal{A}}, a_i^{\mathcal{A}})$ cannot hold (by transitivity and irreflexivity), there must be a new element $a_k \in D_{\mathcal{A}}$, and so on ...

4. (0,5 points) Let us denote by pLJ$^+$ the sequent calculus pLJ *extended with* the axiom (schema) $\vdash A \vee \neg A$. (a) Is the sequent $\vdash \neg\neg A \to A$ derivable in pLJ$^+$? (in the affirmative case show the derivation). (b) Is $\vdash \neg\neg A \to A$ derivable in pLJ$^+$ without using the (cut) rule? (Explain)

   Sketch of solution:

   - easy derivation in pLJ$^+$ using (cut)
   - No. Trying to find a proof (bottom up) of the sequent $\vdash \neg\neg A \to A$ the only rules that can be applied to $\neg\neg A \vdash A$ are weakening right, left or contraction left. None leads to a derivation. The claim follows by the cut-elimination theorem (that holds in LJ)

   Most common mistake: prove $\vdash \neg\neg A \to A$ without using (cut), by applying the $(\neg, l)$ rule of LK, which leads to a sequent $\vdash \neg A, A$ that is *not* allowed in pLJ$^+$!!!!

5. (1 point) Exhibit a formula that is satisfiable in *all and only* the interpretations having k elements ($k \geq 2$) in their domains.

   Solution
   Let $\exists^{\geq n}$ be the formula

$$\exists x_1 \dots x_n \bigwedge_{i \neq j} \neg(x_i = x_j)$$

   The required formula is

$$\exists^{\geq n} \wedge \neg\exists^{\geq n+1}$$

   (other formulas are possible)

Computability

1. (3 points) Are the following sets

   (a) $\{i \mid \exists n, \Phi_i(n) \downarrow \text{ and } \Phi_i(n+1) \downarrow\}$

   (b) $\{i \mid Dom(\Phi_i) \cap Dom(\Phi_a) = \emptyset\}$ in case $Dom(\Phi_a) \neq \emptyset$

   (c) $\{i \mid Dom(\Phi_i) \cap Dom(\Phi_a) = \emptyset\}$ in case $Dom(\Phi_a) = \emptyset$

   recursive, r.e. or none of them? (Prove your claims)

   Sketch of solutions:

   - Not recursive (show that the set is $\neq \emptyset$ and $\neq \mathcal{N}$ and use Rice's theorem), recursively enumerable (use the dovetailing technique)
   - Not recursive (use Rice's theorem), not recursively enumerable (use Post's theorem after having shown, using the *dovetailing technique*, that the complementar set is recursively enumerable)
   - Recursive (the set is equal to $\mathcal{N}$)

2. (1 point) Prove that there is an index p such that $\Phi_p(0) = p^2$?

Sketch of solution:

(very similar to the example in p. 18, computability slide nr. 4)

use the auxiliary function

$$f(x, y) = x^2$$

the smn theorem and the fixed point theorem.

3. (1 point) Exhibit a lambda term which simulates the boolean function "$\leftrightarrow$" (i.e. $A \leftrightarrow B$ is true if and only if either $A = B = \mathbf{T}$ or $A = B = \mathbf{F}$) (hint: encode true $\mathbf{T}$ by $\lambda xy.x$ and false $\mathbf{F}$ by $\lambda xy.y$)

Solution: various possibilities. For instance

$$\lambda xy.xy(y\mathbf{F}\mathbf{T})$$