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Exercises Block 3 - Incompleteness
6.0 VU Logic and Computability
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Exercise IC1 *Prove that, if Σ is correct and $\overline{P}^*$ is expressible, then Σ is Gödel-incomplete.*

If $\overline{P}^*$ is expressible, then there must be a predicate $H \in \mathcal{H}$ s.t. $n \in \overline{P}^* \iff H(n) \in \mathcal{T}$ for all $n \in \mathcal{N}$. Now, let $h = \ulcorner H \urcorner$ and G be the diagonalization of H, i.e. $G = H(h)$. By the definition of $\overline{P}^*$, we get $n \in \overline{P}^* \iff \ulcorner E_n(n) \urcorner \in \overline{P}$ for all $n \in \mathcal{N}$. Therefore $h \in \overline{P}^* \iff \ulcorner H(h) \urcorner \in \overline{P} \iff \ulcorner G \urcorner \in \overline{P} \iff \ulcorner G \urcorner \notin P \iff G \notin \mathcal{P}$. Since $\overline{P}^*$ is expressible we have $h \in \overline{P}^* \iff H(h) \in \mathcal{T} \iff G \in \mathcal{T}$, and furthermore $G \in \mathcal{T} \iff G \notin \mathcal{P}$. Since Σ is assumed to be correct it cannot be the case that $G \notin \mathcal{T}$ and $G \in \mathcal{P}$, therefore $G \in \mathcal{T}$ and $G \notin \mathcal{P}$. Moreover, since Σ is correct we have that $G \notin \mathcal{R}$. Hence, $G \notin \mathcal{P}$ and $G \notin \mathcal{R}$, therefore G is undecidable and Σ is Gödel-incomplete.

Exercise IC2 *Provide an example of an expressible, but not representable set.*

We will consider the set $\overline{P}^*$, in class we showed that $\overline{P}^*$ is expressible in Formal Arithmetic, and in the following we will show that it is not representable in any system.

Assume $\overline{P}^*$ is representable, then there is a predicate $H \in \mathcal{H}$ s.t. $n \in \overline{P}^* \iff H(n) \in \mathcal{P}$. Additionally, by the definition of $\overline{P}^*$ we get $n \in \overline{P}^* \iff \ulcorner E_n(n) \urcorner \in \overline{P} \iff E_n(n) \notin \mathcal{P}$ for all $n \in \mathcal{N}$. More specifically, it must also hold for $h = \ulcorner H \urcorner$, i.e. $H(h) \in \mathcal{P} \iff E_h(h) \notin \mathcal{P} \iff H(h) \notin \mathcal{P}$, which is a contradiction. Hence, $\overline{P}^*$ is not representable in any system Σ.

Exercise IC3 *What follows from the incompleteness theorems about the provability of partial correctness assertions? What if A, B, and/or π are trivial? Provide concrete examples and explain!*

Consider the partial correctness assertions $\{\textbf{true}\}\texttt{skip}\{A\}$ and $\{A\}\texttt{skip}\{\textbf{false}\}$. Both assertions only depend on A, i.e. proving those assertions would give us a decision procedure for A. This contradicts the Incompleteness Theorem.