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Exercises Block 2
6.0 VU Logic and Computability
2014W

Exercise ML2 *For each of the following schematic formulas, show that they are valid in $\mathcal{F}$ and provide concrete counter-examples that show that removal of some accessibilities leads to invalidity:*

$\Box(A \supset A)$

Since $A \supset A$ is a classical tautology, it holds in every world, and therefore $\Box(A \supset A)$ holds in every world as well. Furthermore, the removal of some accessibilities does not yield invalidity.

$\Diamond\Diamond A \supset \Diamond A$

Let $x \in \{w, u\}$ be an arbitrary world and A an arbitrary formula, and suppose $\mathcal{M}, x \models \Diamond\Diamond A$. Then, there exists a world $y \in \{w, u\}$ s.t. xRy and $\mathcal{M}, y \models \Diamond A$, moreover there must be a world $z \in \{w, u\}$ s.t. yRz and $\mathcal{M}, z \models A$. Since $\mathcal{F}$ is transitive, xRz must hold as well, and therefore $\mathcal{M}, x \models \Diamond A$, and $\mathcal{M}, x \models \Diamond\Diamond A \supset \Diamond A$. We can conclude $\mathcal{F} \models \Diamond\Diamond A \supset \Diamond A$.

The removal of wRw yields invalidity. Consider a model $\mathcal{M}$ based on $\mathcal{F}$ s.t. $v_{\mathcal{M}}(p, w) = \mathbf{1}$ and $v_{\mathcal{M}}(p, u) = \mathbf{0}$. Then $\mathcal{M}, w \models \Diamond\Diamond p$ holds, but $\mathcal{M}, w \not\models \Diamond p$. Therefore $\mathcal{M}, w \not\models \Diamond\Diamond p \supset \Diamond p$ and $\mathcal{F} \not\models \Diamond\Diamond p \supset \Diamond p$.

$A \supset \Box\Diamond A$

Let $x \in \{w, u\}$ be an arbitrary world and A an arbitrary formula, and suppose $\mathcal{M}, x \models A$. Since $\mathcal{F}$ is symmetric we have $\mathcal{M}, y \models \Diamond A$ for each world $y \in \{w, u\}$ s.t. xRy. Therefore we have $\mathcal{M}, x \models \Box\Diamond A$ and $\mathcal{M}, x \models A \supset \Box\Diamond A$. We can conclude $\mathcal{F} \models A \supset \Box\Diamond A$.

The removal of uRw yields invalidity. Consider a model $\mathcal{M}$ based on $\mathcal{F}$ s.t. $v_{\mathcal{M}}(p, w) = \mathbf{1}$ and $v_{\mathcal{M}}(p, u) = \mathbf{0}$. Then $\mathcal{M}, w \models p$ holds, and $\mathcal{M}, u \not\models \Diamond p$. Therefore $\mathcal{M}, w \not\models \Box\Diamond p$ and $\mathcal{M}, w \not\models p \supset \Box\Diamond p$. Moreover $\mathcal{F} \not\models p \supset \Box\Diamond p$.

Exercise ML3 *Show that the intersection of two logics is also a logic. What about unions of logics? (Prove or refute!)*

Let $\mathcal{L}_1, \mathcal{L}_2$ be arbitrary logics and $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$ the intersection of them. Furthermore, let $F, F \supset G \in \mathcal{L}$. Hence, $F, F \supset G \in \mathcal{L}_1$ and $F, F \supset G \in \mathcal{L}_2$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed under *modus ponens*, we have $G \in \mathcal{L}_1$, $G \in \mathcal{L}_2$, and therefore $G \in \mathcal{L}$. In conclusion, if $F, F \supset G \in \mathcal{L}$ then $G \in \mathcal{L}$, therefore $\mathcal{L}$ is closed under *modus ponens*.

Let $F \in \mathcal{L}$ be an arbitrary formula, π an arbitrary substitution ($PV_{\mathcal{L}} \mapsto FORM_{\mathcal{L}}$) and $F[\pi]$ the formula resulting from F by applying π. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed under *substitution*, we have $F[\pi] \in \mathcal{L}_1$, $F[\pi] \in \mathcal{L}_2$, and therefore $F[\pi] \in \mathcal{L}$. Hence, we can conclude $\mathcal{L}$ is closed under *substitution*.

Let $\mathcal{L}_1 = \{\bot\}$ and $\mathcal{L}_2 = \{\bot \supset \top\}$. Both, $\mathcal{L}_1$ and $\mathcal{L}_2$, are logics since both are closed under *substitution* and *modus ponens*. However, the union $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 = \{\bot, \bot \supset \top\}$ is not closed under *modus ponens* since $\top \notin \mathcal{L}$, and therefore it is not a logic.

Exercise ML4 *Specify a concrete counter-example to $F \supset \Box F$. Is there a frame in which $F \supset \Box F$ is valid? (Prove or refute!)*

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model s.t. there are worlds $w, u \in W$ and wRu. Now consider an assignment $v_{\mathcal{M}}(p, w) = \mathbf{1}$ for some $p \in PV$ and $v_{\mathcal{M}}(p, x) = \mathbf{0}$ for all $x \in W$ different from w. Then $\mathcal{M}, w \models p$ and $\mathcal{M}, w \not\models \Box p$, and therefore $\mathcal{M}, w \not\models p \supset \Box p$.

Let $\mathcal{F} = \langle W, \emptyset \rangle$ be a frame. Then, trivially $\mathcal{F} \models F \supset \Box F$ holds, since $\Box F$ holds in every world $w \in W$ and every model $\mathcal{M}$ based on $\mathcal{F}$.

Exercise I1 *Show that formulas 3, 6, 7 and 9 (on slide 8) are BHK-valid.*

(3) $(A \wedge B) \supset (B \wedge A)$

The procedure $\gamma \rhd \{(A \wedge B) \supset (B \wedge A)\}$ transforms every proof $\rho = \langle \rho_A, \rho_B \rangle$ of $(A \wedge B)$ into a proof of $(B \wedge A)$, where ρ_A is a proof of A and ρ_B is a proof of B. The procedure γ can be described as follows:

1. extract the first component ρ_A from ρ
2. extract the second component ρ_B from ρ
3. return $\langle \rho_B, \rho_A \rangle$

(6) $\bot \supset A$

The procedure $\gamma \rhd \{\bot \supset A\}$ takes as input a proof of $\bot$ and returns a proof of A. Trivially, γ transforms any given proof of $\bot$ into a proof of A, since there are no proofs for $\bot$.

(7) $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$

We define the procedure $\gamma \rhd \{(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)\}$ as follows: The input of γ is a procedure $\eta \rhd \{A \supset (B \supset C)\}$ where the

- input of η is a proof $\delta \rhd \{A\}$, and the
- output of η is a procedure $\pi \rhd \{B \supset C\}$.

The output of γ, i.e. the procedure ν – which transforms a proof $\rho = \langle \rho_A, \rho_B \rangle$ into a proof σ of C – can be described as follows:

1. extract the first component ρ_A from ρ
2. apply η to ρ_A, i.e. compute $\eta(\rho_A)$, we get a proof π of $B \supset C$

3. extract the second component ρ_B from ρ
4. apply π to ρ_B, i.e. compute $\pi(\rho_B)$, we get a proof σ of C
5. return σ

(9) $(A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C))$

Consider the procedure $\gamma \triangleright \{(A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C))\}$. The input of γ is a proof $\sigma = \langle \alpha, \rho \rangle$ s.t. $\alpha \triangleright \{A\}$ and $\rho = \langle \nu, \rho_0 \rangle \triangleright \{B \vee C\}$. The procedure γ can be described as follows:

1. extract the first component – i.e. α – from σ
2. extract the second component – i.e. $\rho = \langle \nu, \rho_0 \rangle$ – from σ
3. extract the first component – i.e. ν – from ρ
4. extract the second component – i.e. ρ_0 – from ρ
5. create the pair $\tau = \langle \alpha, \rho_0 \rangle$
6. return the pair $\langle \nu, \tau \rangle$

Exercise AD1 *Show the validity of the following formula in LK:* $sk(\forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)) \rightarrow \forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)$.

$$sk(\forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)) = \forall x \forall y \forall u F(x, y, f(x, y), u, g(x, y, u))$$

$$\begin{array}{rcll}
F(a, b, f(a, b), c, g(a, b, c)) & \vdash & F(a, b, f(a, b), c, g(a, b, c)) & \\ \hline
F(a, b, f(a, b), c, g(a, b, c)) & \vdash & \exists v F(a, b, f(a, b), c, v) & \exists, r \\ \hline
\forall u F(a, b, f(a, b), u, g(a, b, u)) & \vdash & \exists v F(a, b, f(a, b), c, v) & \forall, l \\ \hline
\forall u F(a, b, f(a, b), u, g(a, b, u)) & \vdash & \forall u \exists v F(a, b, f(a, b), u, v) & \forall, r \\ \hline
\forall u F(a, b, f(a, b), u, g(a, b, u)) & \vdash & \exists z \forall u \exists v F(a, b, z, u, v) & \exists, r \\ \hline
\forall y \forall u F(a, y, f(a, y), u, g(a, y, u)) & \vdash & \exists z \forall u \exists v F(a, b, z, u, v) & \forall, l \\ \hline
\forall y \forall u F(a, y, f(a, y), u, g(a, y, u)) & \vdash & \forall y \exists z \forall u \exists v F(a, y, z, u, v) & \forall, r \\ \hline
\forall x \forall y \forall u F(x, y, f(x, y), u, g(x, y, u)) & \vdash & \forall y \exists z \forall u \exists v F(a, y, z, u, v) & \forall, l \\ \hline
\forall x \forall y \forall u F(x, y, f(x, y), u, g(x, y, u)) & \vdash & \forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v) & \forall, r \\ \hline
& \vdash & \forall x \forall y \forall u F(x, y, f(x, y), u, g(x, y, u)) \rightarrow \forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v) & \rightarrow, r
\end{array}$$

Remark: a, b and c are free variables.

Why is the inverse implication not valid?

Assuming $\forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)$ holds, then one cannot conclude that $\forall x \forall y \forall u F(x, y, f(x, y), u, g(x, y, u))$ holds for every function f and g. Consider an interpretation $I = (D, \phi, d)$ as counter-example, where

$$D = \{0, 1\}$$
$$\phi(f)(x, y) = 1 \text{ for all } x, y \in D$$
$$\phi(g)(x, y, z) = 1 \text{ for all } x, y, z \in D$$
$$\phi(F)(x, y, z, u, v) = \begin{cases} \mathbf{1} & \text{if } z = v = 0 \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Then, $\forall x \forall y \exists z \forall u \exists v F(x, y, z, u, v)$ evaluates to $\mathbf{1}$ and $\forall x \forall y \forall u F(x, y, f(x, y), u, g(x, y, u))$ evaluates to $\mathbf{0}$. Hence, the inverse implication is not valid.

Exercise AD2 *Prove that if $\forall x_1 \cdots \forall x_n F$ is satisfiable, then $\forall x_1 \cdots \forall x_n \delta(F)$ is satisfiable.*

By the lemma about the definition we get that $\forall x_1 \cdots \forall x_n (\epsilon(F) \wedge p_F(x_1, \ldots, x_n))$ is satisfiable iff $\forall x_1 \cdots \forall x_n F$ is satisfiable. It suffices to show that if $\epsilon(F)$ is satisfiable, then $\gamma(F) := \bigwedge_{G \in \Sigma(F)} D_G$ is satisfiable. We will perform an inductive proof on the logical complexity of F.

Basis F is atomic. Then, $\epsilon(F) = E_F$ and E_F has the form $\forall X(p_F(X) \leftrightarrow F)$, by applying equivalence transformations we get

$$\begin{aligned} & \forall X(p_F(X) \leftrightarrow F) \\ \iff & \forall X((p_F(X) \rightarrow F) \wedge (F \rightarrow p_F(X))) \\ \iff & \forall X((\neg p_F(X) \vee F) \wedge (\neg F \vee p_F(X))) \\ \iff & \forall X(\neg p_F(X) \vee F) \wedge \forall X(\neg F \vee p_F(X)). \end{aligned}$$

It is obvious that the last formula is equivalent to D_F. Since $\gamma(F) = D_F$, if $\epsilon(F)$ is satisfiable then $\gamma(F)$ is satisfiable.

Induction hypothesis Assume that, for all H with $lc(H) < m$, if $\epsilon(H)$ is satisfiable, then $\gamma(H)$ is satisfiable.

Step Consider a formula G with $lc(G) = m$. We perform a case distinction wrt. the top-level symbol in G.

Case G is of the form $\neg H$. Then, E_G is of the form $\forall X(p_G(X) \leftrightarrow \neg p_H(X))$, by applying equivalence transformations we get

$$\begin{aligned}
&\forall X(p_G(X) \leftrightarrow \neg p_H(X))\\
\iff\ &\forall X((p_G(X) \rightarrow \neg p_H(X)) \wedge (\neg p_H(X) \rightarrow p_G(X)))\\
\iff\ &\forall X((\neg p_G(X) \vee \neg p_H(X)) \wedge (p_G(X) \vee p_H(X)))\\
\iff\ &\forall X(\neg p_G(X) \vee \neg p_H(X)) \wedge \forall X(p_G(X) \vee p_H(X)).
\end{aligned}$$

Thus, E_G is equivalent to D_G and furthermore if E_G is satisfiable then D_G is satisfiable. Hence, by the induction hypothesis, the equivalence of E_G and D_G, and the construction of D_G, we get that if $\epsilon(G)$ is satisfiable, then $\gamma(G)$ is satisfiable.

Case G is of the form $H_1 \wedge H_2$. Then, E_G is of the form $\forall X(p_G(X) \leftrightarrow (p_{H_1}(Y) \wedge p_{H_2}(Z)))$ where $X = Y \cup Z$, by applying equivalence transformations we get

$$\begin{aligned}
&\forall X(p_G(X) \leftrightarrow (p_{H_1}(Y) \wedge p_{H_2}(Z)))\\
\iff\ &\forall X((p_G(X) \rightarrow (p_{H_1}(Y) \wedge p_{H_2}(Z))) \wedge ((p_{H_1}(Y) \wedge p_{H_2}(Z)) \rightarrow p_G(X)))\\
\iff\ &\forall X((\neg p_G(X) \vee (p_{H_1}(Y) \wedge p_{H_2}(Z))) \wedge (\neg(p_{H_1}(Y) \wedge p_{H_2}(Z)) \vee p_G(X)))\\
\iff\ &\forall X((\neg p_G(X) \vee p_{H_1}(Y)) \wedge (\neg p_G(X) \vee p_{H_2}(Z)) \wedge (\neg p_{H_1}(Y) \vee \neg p_{H_2}(Z) \vee p_G(X)))\\
\iff\ &\forall X(\neg p_G(X) \vee p_{H_1}(Y)) \wedge \forall X(\neg p_G(X) \vee p_{H_2}(Z)) \wedge\\
&\forall X(\neg p_{H_1}(Y) \vee \neg p_{H_2}(Z) \vee p_G(X)). \qquad (1)
\end{aligned}$$

Furthermore D_G is of the form

$$\forall X(\neg p_H(X) \vee p_{H_1}(X)) \wedge \forall X(\neg p_G(X) \vee p_{G_2}(X)) \wedge \forall X(p_G(X) \vee \neg p_{G_1}(X) \vee \neg p_{G_2}(X)). \quad (2)$$

It is obvious that (1) is equivalent to (2). Hence, by the induction hypothesis, the equivalence of (1) and (2), and the construction of D_G, we get that if $\epsilon(G)$ is satisfiable, then $\gamma(G)$ is satisfiable.

Case G is of the form $H_1 \circ H_2$ st. $\circ \in \{\vee, \rightarrow\}$. These cases are analogous to the one above.

Case G is of the form $\forall v H$. Then, E_G is of the form $\forall X(p_G(X) \leftrightarrow \forall v\ p_H(X, v))$, by applying equivalences we get

$$\begin{aligned}
&\forall X(p_G(X) \leftrightarrow \forall v\ p_H(X, v))\\
\iff\ &\forall X((p_G(X) \rightarrow \forall v\ p_H(X, v)) \wedge (\forall v\ p_H(X, v) \rightarrow p_G(X)))\\
\iff\ &\forall X((\neg p_G(X) \vee \forall v\ p_H(X, v)) \wedge (\neg \forall v\ p_H(X, v) \vee p_G(X)))\\
\iff\ &\forall X((\neg p_G(X) \vee \forall v\ p_H(X, v)) \wedge (\exists v\ \neg p_H(X, v) \vee p_G(X)))\\
\iff\ &\forall X \forall v(\neg p_G(X) \vee\ p_H(X, v)) \wedge \forall X(\exists v\ \neg p_H(X, v) \vee p_G(X))
\end{aligned}$$

Considering the skolem form of the last formula

$$\forall X \forall v(\neg p_G(X) \vee \ p_H(X,v)) \wedge \forall X(\neg p_H(X, f(X)) \vee p_G(X)),$$

we obtain a formula which is satisfiable iff E_G is satisfiable. Furthermore it is easy to see that if this skolem form is satisfiable, then so is D_G.

Case G is of the form $\exists v H$. This case is analogous to the one above.

Exercise AD3 *Compute* $\delta(F)'$ *for* $F = \exists y(p(x, g(x,y)) \rightarrow \exists z \neg(p(g(x,z),y) \vee p(y,x)))$.

$$\begin{aligned}
D_\vee: \quad & \forall xyz(\neg p_\vee(x,y,z) \vee p(g(x,z),y) \vee p(y,x)) \wedge \\
& \forall xyz(p_\vee(x,y,z) \vee \neg p(g(x,z),y)) \wedge \\
& \forall xyz(p_\vee(x,y,z) \vee \neg p(y,x)) \\
D_\neg: \quad & \forall xyz(\neg p_\neg(x,y,z) \vee \neg p_\vee(x,y,z)) \wedge \\
& \forall xyz(p_\neg(x,y,z) \vee p_\vee(x,y,z)) \\
D_{\exists_1}: \quad & \forall xy(\neg p_{\exists_1}(x,y) \vee p_\neg(x,y,f(x,y))) \wedge \\
& \forall xy \forall z(p_{\exists_1}(x,y) \vee \neg p_\neg(x,y,z)) \\
D_\rightarrow: \quad & \forall xy(\neg p_\rightarrow(x,y) \vee \neg p(x,g(x,y)) \vee p_{\exists_1}(x,y)) \wedge \\
& \forall xy(p_\rightarrow(x,y) \vee p(x,g(x,y))) \wedge \\
& \forall xy(p_\rightarrow(x,y) \vee \neg p_{\exists_1}(x,y)) \\
D_{\exists_2}: \quad & \forall x(\neg p_{\exists_2}(x) \vee p_\rightarrow(x,h(x))) \wedge \\
& \forall x \forall y(p_{\exists_2}(x) \vee \neg p_\rightarrow(x,y))
\end{aligned}$$

Considering $\delta(F)'$ as a set of clauses we get

$$\begin{aligned}
\{&\neg p_\vee(x,y,z) \vee p(g(x,z),y) \vee p(y,x), p_\vee(x,y,z) \vee \neg p(g(x,z),y), p_\vee(x,y,z) \vee \neg p(y,x), \\
&\neg p_\neg(x,y,z) \vee \neg p_\vee(x,y,z), p_\neg(x,y,z) \vee p_\vee(x,y,z), \\
&\neg p_{\exists_1}(x,y) \vee p_\neg(x,y,f(x,y)), p_{\exists_1}(x,y) \vee \neg p_\neg(x,y,u), \\
&\neg p_\rightarrow(x,y) \vee \neg p(x,g(x,y)) \vee p_{\exists_1}(x,y), p_\rightarrow(x,y) \vee p(x,g(x,y)), p_\rightarrow(x,y) \vee \neg p_{\exists_1}(x,y), \\
&\neg p_{\exists_2}(x) \vee p_\rightarrow(x,h(x)), p_{\exists_2}(x) \vee \neg p_\rightarrow(x,u), p_{\exists_2}(x)\}.
\end{aligned}$$

Exercise AD4 *Find all Robinson-resolvents of* $C = p(x, f(x)) \vee p(a,y)$ *and* $D = \neg p(x,y) \vee \neg p(a, f(x)) \vee \neg p(f(x), f(y))$. *Specify all used renamings, mgus and (implicit) factors.*

As a first step, we only consider Robinson-resolvents of trivial factors, and we consider the variable disjoint variant $C' = p(u, f(u)) \lor p(a, v)$ with $\nu = \{x \mapsto u, y \mapsto v\}$.

- Resolving upon first literal of C' and first literal of D using mgu $\{x \mapsto u, y \mapsto f(u)\}$ we obtain the resolvent $p(a, v) \lor \neg p(a, f(u)) \lor \neg p(f(u), f(f(u)))$.
- Resolving upon first literal of C' and second literal of D using mgu $\{u \mapsto a, x \mapsto a\}$ we obtain the resolvent $p(a, v) \lor \neg p(a, y) \lor \neg p(f(a), f(y))$.
- Resolving upon first literal of C' and third literal of D using mgu $\{u \mapsto f(x), y \mapsto f(x)\}$ we obtain the resolvent $p(a, v) \lor \neg p(x, f(x)) \lor \neg p(a, f(x))$.
- Resolving upon second literal of C' and first literal of D using mgu $\{x \mapsto a, y \mapsto v\}$ we obtain the resolvent $p(u, f(u)) \lor \neg p(a, f(a)) \lor \neg p(f(a), f(v))$.
- Resolving upon second literal of C' and second literal of D using mgu $\{v \mapsto f(x)\}$ we obtain the resolvent $p(u, f(u)) \lor \neg p(x, y) \lor \neg p(f(x), f(y))$.

In the next step, we consider Robinson-resolvents of non-trivial factors.

- Let $C' = p(a, f(a))$ be a factor of C by using the mgu $\{x \mapsto a, y \mapsto f(a)\}$. Resolving upon the first literal of C' and the first literal of D using mgu $\{x \mapsto a, y \mapsto f(a)\}$ yields the resolvent $\neg p(a, f(a)) \lor \neg p(f(a), f(f(a)))$.
- Let $C' = p(a, f(a))$ be a factor of C by using the mgu $\{x \mapsto a, y \mapsto f(a)\}$. Resolving upon the first literal of C' and the second literal of D using mgu $\{x \mapsto a\}$ yields the resolvent $\neg p(a, y) \lor \neg p(f(a), f(y))$.
- Let $D' = \neg p(a, f(a)) \lor \neg p(f(a), f(f(a)))$ be a factor of D by using the mgu $\{x \mapsto a, y \mapsto f(a)\}$. Resolving upon the first literal of C and the first literal of D' using mgu $\{x \mapsto a\}$ yields the resolvent $p(a, y) \lor \neg p(f(a), f(f(a)))$.
- Let $D' = \neg p(a, f(a)) \lor \neg p(f(a), f(f(a)))$ be a factor of D by using the mgu $\{x \mapsto a, y \mapsto f(a)\}$. Resolving upon the first literal of C and the second literal of D' using mgu $\{x \mapsto f(a)\}$ yields the resolvent $p(a, y) \lor \neg p(a, f(a))$.
- Let $D' = \neg p(a, f(a)) \lor \neg p(f(a), f(f(a)))$ be a factor of D by using the mgu $\{x \mapsto a, y \mapsto f(a)\}$. Resolving upon the second literal of C and the first literal of D' using mgu $\{y \mapsto f(a)\}$ yields the resolvent $p(x, f(x)) \lor \neg p(f(a), f(f(a)))$.
- Let $C' = p(a, f(a))$ be a factor of C by using the mgu $\{x \mapsto a, y \mapsto f(a)\}$ and $D' = \neg p(a, f(a)) \lor \neg p(f(a), f(f(a)))$ be a factor of D by using the mgu $\{x \mapsto a, y \mapsto f(a)\}$. We can immediately resolve upon the first literal of C' and the first literal of D' and we obtaion the resolvent $\neg p(f(a), f(f(a)))$.

Exercise AD5 *Describe another resolution refinement (other than ordered resolution) in detail. (You do not have to give a completeness proof).*

In the following, we will consider *hyperresolution* as a resolution refinement. The desription is based on the paper about *Resolution Theorem Proving*[1], especially the definition and the example are taken from this work.

Hyperresolution is the deduction principle where only *positive* clauses and the empty clause are derivable. A clause is called *positive* if it is of the form $\vdash A_1, \dots, A_n$, where $A_1, \dots, A_n$ are atoms. The derivation of just positive clauses is only possible if we use many-step instead of one-step inferences.

Definition 1. *Let C be a nonpositive clause and $D_1, \dots, D_n$ be positive clauses; then $S : (C; D_1, \dots, D_n)$ is called a clash sequence. Let $C_0 = 0$ and $C_i + 1 \in \mathcal{R}es(\{C_i, D_{i+1}\})$ for $i = 1, \dots, n-1$. If C_n is defined and positive then it is called a hyperresolvent of S.*

Remark. $\mathcal{R}es(\mathcal{C})$ denotes the set of resolvents definable from a set of clauses $\mathcal{C}$.

Example 1. *Let $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ be a set of clauses where*

$$\begin{aligned} C1 &= \ \vdash p(a,b) \\ C2 &= \ \vdash p(b,a) \\ C3 &= \ p(x,y), p(y,z) \vdash p(x,z) \\ C4 &= \ p(a,a) \vdash . \end{aligned}$$

We can construct a refutation of $\mathcal{C}$:

$$\dfrac{\dfrac{\vdash\ p(a,b) \qquad \dfrac{\vdash\ p(b,a) \qquad p(x,y), p(y,z)\ \vdash\ p(x,z)}{p(x,b)\ \vdash\ p(x,a)}}{\vdash\ p(a,a)} \qquad p(a,a)\ \vdash}{\vdash}$$

Note that $\vdash p(a,a)$ is a hyperresolvent of the clash sequence $(C_3; C_1, C_2)$.

[1] Alexander Leitsch, `http://www.logic.at/staff/leitsch/httpd/resolv.pdf`