VORLESUNG

TECHNISCHE HYDRAULIK

222.564

Exercises

Euler momentum theorem – Bernoulli‘s equation – Open-channel flow

Ex 1. Euler momentum theorem

Input data:

- ρ = 1000 [kg m^{-3}]
- μ = 1.0 x 10^{-3} [kg m^{-1} s^{-1}]
- Q = 8 [m^3 s^{-1}]
- D = 1 [m]
- Δh = 115 [m]
- Δl = 160 [m]
- Penstock in galvanized steel: k_s = 0.15 x 10^{-3} [m]

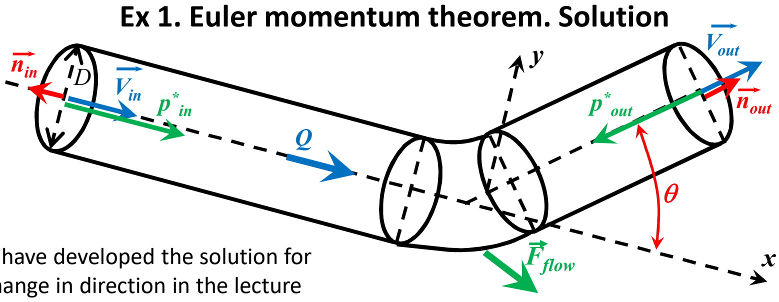
Let us consider again the penstock pipe of the Opponitz power plant. Let us consider again the case detailed in TH_PipeFlow, where a discharge of Q = 8 [m^3 s^{-1}] is obtained by partially closing a valve that is situated at the downstream end of the penstock.

Imagine now that the penstock pipe is not straight, but has a change in direction of 75 [°] near its downstream end. Assume that the last 20 [m] of the penstock, including the bend and the valve, are flat.

Determine the force induced by the flow on the pipe due to this change in direction. Determine the total force, the downslope component of the force, and the transverse component of the force for two configurations: the first with the bend just upstream of the valve and the second with the bend just downstream of the valve. For what configuration is the force smallest ?

We have considered this example already in TH_PipeFlow

Ex 1. Euler momentum theorem. Solution



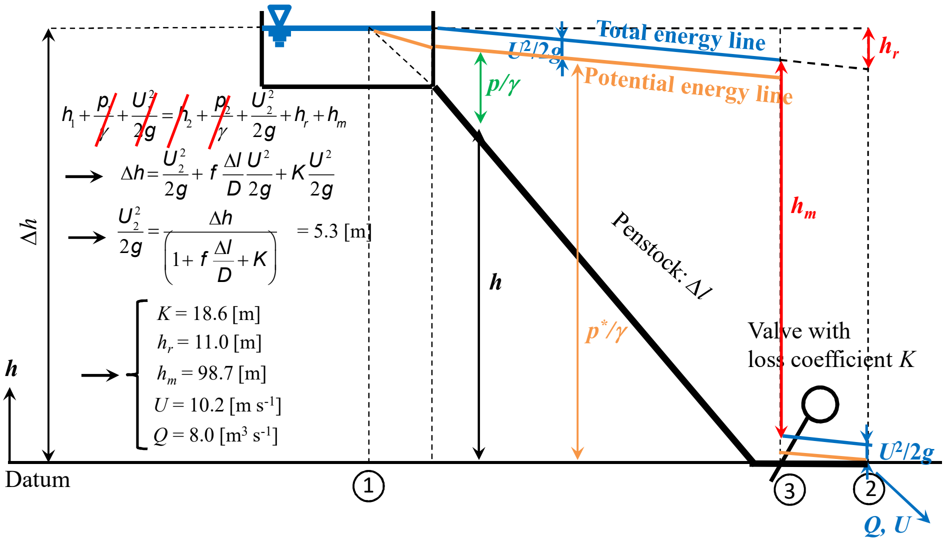
- We have developed the solution for a change in direction in the lecture TH_EulerMomentumTheorem:

$$\oint_S \rho \vec{V}\left(\vec{V}.\vec{n}\right)dS = \cancel{\vec{F}_g} + \vec{F}_p$$

$$\begin{cases} F_{x,flow} = \left(\rho\dfrac{Q^2}{S} + pS\right)(1-\cos\theta) \\ F_{y,flow} = -\left(\rho\dfrac{Q^2}{S} + pS\right)(\sin\theta) \end{cases}$$

- We neglect the weight of the water in the control volume, $\vec{F}_g$
- The only unknown in the equation is p at the location of the bend. It has to be determined by considering the energy budget along the entire system (Bernoulli equation extended with energy losses). This energy budget has been detailed in TH_PipeFlow and is reproduced hereafter. There is only a minor change to account for the horizontality of the last 20 [m] of the pipe.

Ex 1. Euler momentum theorem. Solution



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Ex 1. Euler momentum theorem. Solution

- Assume that the valve is in the middle of the straight reach (10 [m] from the penstock exit) and that the bend is just upstream/downstream of the valve.
- We find p/γ at the bend by applying the Bernoulli equation between a point where we know the flow characteristics and energy level (such as the surface of the reservoir or the outflow section of the penstock) and the bend (point 3 in the figure on the previous slide). Let us choose the surface of the reservoir as known point.

$$h_1 + \cancel{\frac{p_1}{\gamma}} + \cancel{\frac{U_1^2}{2g}} = \cancel{h_3} + \frac{p_3}{\gamma} + \frac{U_3^2}{2g} + h_r \left(+h_m\right)$$

The minor energy losses are written in parentheses because they have to be excluded/included for a bend upstream/downstream of the valve

$$\rightarrow \qquad \frac{p_3}{\gamma} = h_1 - \frac{U^2}{2g} - f\frac{\Delta l}{D}\frac{U^2}{2g}\left(-K\frac{U^2}{2g}\right) = h_1 - \frac{U^2}{2g}\left[1 + f\frac{\Delta l}{D}\left(+K\right)\right]$$

- Δl = 150 [m]
- f = 0.013; value determined for this case in the examples in TH_PipeFlow
- $U^2/2g$ = 5.3 [m]: value given in previous slide
- K = 18.6: value given in previous slide

$$\rightarrow \begin{cases} \dfrac{p_3}{\gamma} = 99.4\ [m] & \text{for a bend upstream of the valve} \\ \dfrac{p_3}{\gamma} = 0.8\ [m] & \text{for a bend downstream of the valve} \end{cases}$$

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Ex 1. Euler momentum theorem. Solution

	Bend upstream of valve	Bend downstream of valve
$\begin{cases} F_{x,flow} = \left(\rho \dfrac{Q^2}{S} + p\, S\right)\left(1-\cos\theta\right) \\ F_{y,flow} = -\left(\rho \dfrac{Q^2}{S} + p\, S\right)\left(\sin\theta\right) \end{cases}$	= (8'149 + 76'585) x 0.74 = 62.8 [kN] = (8'149 + 76'585) x 0.97 = 81.8 [kN]	= (8'149 + 6'164) x 0.74 = 10.6 [kN] = (8'149 + 6'164) x 0.97 = 13.8 [kN]

Remarks:

- The forces exerted by the flow on the penstock in the bend, and hence forces required to anchor the penstock bend, are much smaller when the bend is installed downstream of the valve.
- The second term involving p can be more important than the first involving the momentum flux. This remark cannot be generalized for all cases.

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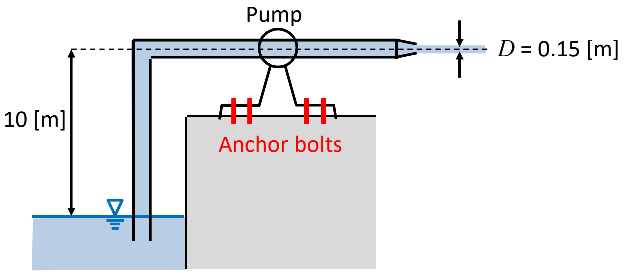
Ex 2. Euler momentum theorem

A horizontal jet is generated by pumping water out of a large reservoir. The discharge of the jet is Q = 0.1 [m^3 s^{-1}] and the jet diameter is D = 0.15 [m]. The free surface of the reservoir is situated 10 [m] below the jet axis. Energy losses in the system can be neglected.



1. Determine the energy head introduced by the pump in the system.
2. How many bolts are required to anchor the pump, if one bolt has an admissible shear force of 25 [N] ?

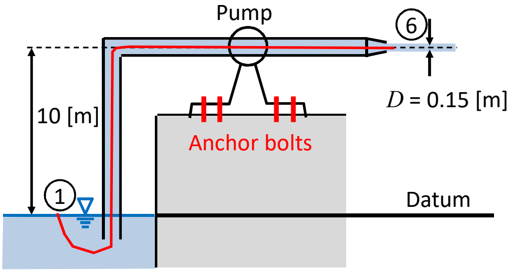
Ex 2. Euler momentum theorem. Solution

1. **Determine the energy head introduced by the pump in the system.**

The energy head introduced by the pump is determined by applying the Bernoulli equation along a streamline (shown in red) between point 1 at the free surface of the reservoir and point 6 in the jet at the pipe outlet. The Bernoulli equation includes the term h_{pump} that represents the energy head introduced by the pump.

$$h_1 + \frac{p_1}{\gamma} + \frac{U_1^2}{2g} + h_{pump} = h_6 + \frac{p_6}{\gamma} + \frac{U_6^2}{2g}$$

- Pressure is equal to atmospheric pressure at the free surface and in the jet ($p_1 = p_6 = p_a = 0$)
- $U_1 = 0$



$$\rightarrow h_{pump} = h_6 - h_1 + \frac{U_6^2}{2g} = h_6 - h_1 + \frac{8Q^2}{g\pi^2 D^4} = 11.6\ [\mathrm{m}]$$

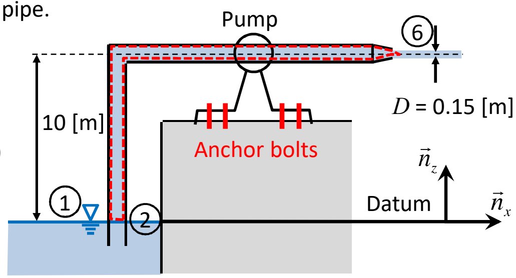
The pump has to add 10 [m] of potential energy and 1.6 [m] of kinetic energy.

- Note that another important characteristic is the required hydraulic power of the pump. It is given by $\mathcal{P} = \gamma Q h_{pump}$ [W]. In the present example, a pump with a hydraulic power of $\mathcal{P}$ = 11.4 [kW] would be required.
- Note that this design will induce a cavitation risk.

Ex 2. Euler momentum theorem. Solution

2. How many bolts are required to anchor the pump, if one bolt has an admissible shear force of 25 [N] ?

The anchorage of the pump has to balance the force exerted by the flow on the wall of the entire pipe system. Because the flow is frictionless, this force is the integral of the pressure forces on the pipe wall. Therefore, we choose a control volume (in red in the figure) that encompasses the entire pipe.



$$\oint_S \rho \vec{V}\left(\vec{V}.\vec{n}\right)dS = \vec{F}_g + \vec{F}_p + \cancel{\vec{F}_\sigma}$$

(See TH_EulerMomentumTheorem)

$\vec{F}_\sigma = 0$ in frictionless flow

$\vec{F}_g = -\gamma V \vec{n}_z$ is the weight of the fluid in the control volume V

$\vec{F}_p = p_2 S_2 \vec{n}_z - p_6 S_6 \vec{n}_x - \vec{F}_{anchor}$ Where $\vec{F}_{anchor}$ is the force exerted by the flow on the anchorage, which is equal in magnitude but opposite in sign to the force exerted by the pipe wall on the flow.

Assume for the sake of simplicity that the pipe has a constant cross-section S_{pipe}

Ex 2. Euler momentum theorem. Solution

$$\rightarrow \vec{F}_{anchor} = -\gamma V\vec{n}_z - \oint_{S2} \rho\vec{V}\left(\vec{V}.\vec{n}\right)dS - \oint_{S6} \rho\vec{V}\left(\vec{V}.\vec{n}\right)dS + p_2 S_2 \vec{n}_z - p_6 S_6 \vec{n}_x$$

$$\rightarrow \begin{cases} F_{x,anchor} = 0 - 0 - \dfrac{\rho Q^2}{S_{nozzle}} + 0 \quad \cancel{-p_a S_{nozzle}} \quad = 566 \text{ [N]} \\ F_{z,anchor} = -\gamma V + \dfrac{\rho Q^2}{S_{pipe}} - 0 + p_2 S_{pipe} - 0 \end{cases}$$

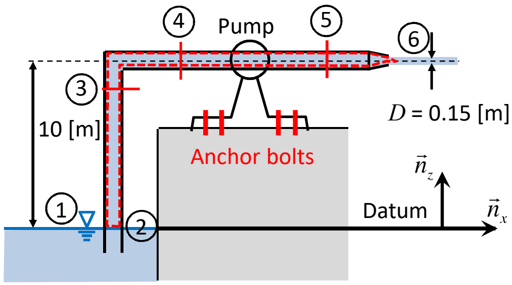
→ 23 bolts are required to anchor the pipe

This example is not as simple and trivial as it might look like at first sight. It is appropriate to illustrate the importance of the choice of the control volume. Let us analyse the different components of this anchor force by dividing the control volume in different parts.

The total force $\vec{F}_{anchor}$ comes from four contributions:

- $\vec{F}_{vert}$ between sections 2 and 3
- $\vec{F}_{bend}$ between sections 3 and 4
- $\vec{F}_{pump}$ between sections 4 and 5
- $\vec{F}_{nozzle}$ between sections 5 and 6

These four contributions will now be analysed.



Ex 2. Euler momentum theorem. Solution

1. The vertical pipe:

$$\vec{F}_{vert} = -\gamma V_{vert}\vec{n}_z + \frac{\rho Q^2}{S_{pipe}}\vec{n}_z - \frac{\rho Q^2}{S_{pipe}}\vec{n}_z + p_2 S_{pipe}\vec{n}_z - p_3 S_{pipe}\vec{n}_z \rightarrow \begin{cases} F_{x,vert} = 0 + 0 - 0 + 0 - 0 \\ F_{z,vert} = -\gamma V_{vert} + \frac{\rho Q^2}{S_{pipe}} - \frac{\rho Q^2}{S_{pipe}} + p_2 S_{pipe} - p_3 S_{pipe} \end{cases}$$

2. The bend:

$$\vec{F}_{bend} = -\gamma V_{bend}\vec{n}_z + \frac{\rho Q^2}{S_{pipe}}\vec{n}_z - \frac{\rho Q^2}{S_{pipe}}\vec{n}_x + p_3 S_{pipe}\vec{n}_z - p_4 S_{pipe}\vec{n}_x \rightarrow \begin{cases} F_{x,bend} = 0 - 0 - \frac{\rho Q^2}{S_{pipe}} + 0 - p_4 S_{pipe} \\ F_{z,bend} = -\gamma V_{bend} + \frac{\rho Q^2}{S_{pipe}} - 0 + p_3 S_{pipe} - 0 \end{cases}$$

3. The pump:

$$\vec{F}_{pump} = -\gamma V_{pump}\vec{n}_z + \frac{\rho Q^2}{S_{pipe}}\vec{n}_x - \frac{\rho Q^2}{S_{pipe}}\vec{n}_x + p_4 S_{pipe}\vec{n}_x - p_5 S_{pipe}\vec{n}_x \rightarrow \begin{cases} F_{x,pump} = 0 + \frac{\rho Q^2}{S_{pipe}} - \frac{\rho Q^2}{S_{pipe}} + p_4 S_{pipe} - p_5 S_{pipe} \\ F_{z,pump} = -\gamma V_{pump} + 0 - 0 + 0 - 0 \end{cases}$$

4. The nozzle:

$$\vec{F}_{nozzle} = -\gamma V_{nozzle}\vec{n}_z + \frac{\rho Q^2}{S_{pipe}}\vec{n}_x - \frac{\rho Q^2}{S_{nozzle}}\vec{n}_x + p_5 S_{pipe}\vec{n}_x - p_6 S_{nozzle}\vec{n}_x \rightarrow \begin{cases} F_{x,nozzle} = 0 + \frac{\rho Q^2}{S_{pipe}} - \frac{\rho Q^2}{S_{nozzle}} + p_5 S_{pipe} - p_6 S_{nozzle} \\ F_{z,nozzle} = -\gamma V_{nozzle} + 0 - 0 + 0 - 0 \end{cases}$$

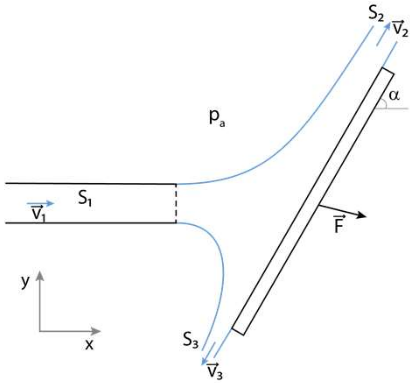
- Where p_2, p_3, p_4, p_5 and p_6 are obtained by applying Bernoulli's equation along a streamline
- The sum of the four contributions gives the total forces derived as answer to question 2.
- An erroneous choice of the control volume that does not encompass the entire pipe wall would only have given part of the force exerted by the flow on the pump anchorage

Ex 3. Euler momentum theorem

Water (ρ = 1000 [kg m^{-3}]) is pumped out of a pipe with cross-sectional area S_1 = 0.2 [m^2] at a velocity of U_1 = 10 [m s^{-1}] before it hits a stationary plate that is tilted at an angle α = 60 [°] relative to the incoming jet. The incoming jet is split into two outgoing jets. Assume that the effects of gravity can be neglected. The jet is surrounded by air at atmospheric pressure. The figure pictures the top view of the jet exiting the pipe and hitting the plate.

1. Prove that the magnitudes of the outgoing velocities U_2 and U_3 have to be equal to the magnitude of the velocity U_1 of the incoming jet using Bernoulli's equation.

2. Choose a control volume and calculate the x- and y-component of the total force F the jet exerts on the plate. Assume that $S_2 = 2S_3$.



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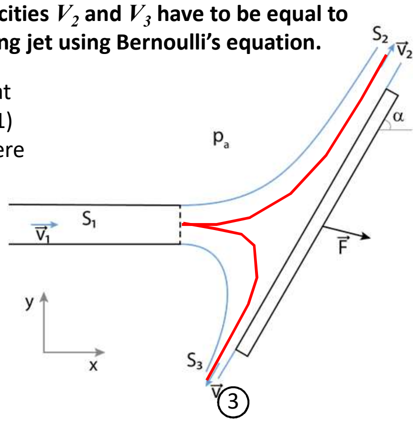
Ex 3. Euler momentum theorem. Solution

1. Prove that the magnitudes of the outgoing velocities V_2 and V_3 have to be equal to the magnitude of the velocity V_1 of the incoming jet using Bernoulli's equation.



- Apply the Bernoulli equation to a streamline that originates in the jet near the pipe outlet (point 1) and one in the jet near the plate in a region where the streamline is straight again (point 2):

$$h_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + h_{pump} = h_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

- The plane is horizontal: $h_1 = h_2$.
- The pressure at the jet-air interface is atmospheric. In sections with quasi-straight and quasi-linear streamlines, the pressure does not vary perpendicularly to the streamline (TH_BernoulliEquation). As a result, $p = p_a$ in the cross-sections containing points 1 and 2.
  → $V_1 = V_2$
- Note that in general when considering high-velocity jets (also non-horizontal ones) h_1-h_2 is negligible with respect to $V^2/2g$, which implies that $V_1 \approx V_2$.
- By applying Bernoulli's equation to a streamline between points 1 and 3, on finds $V_1 = V_3$.

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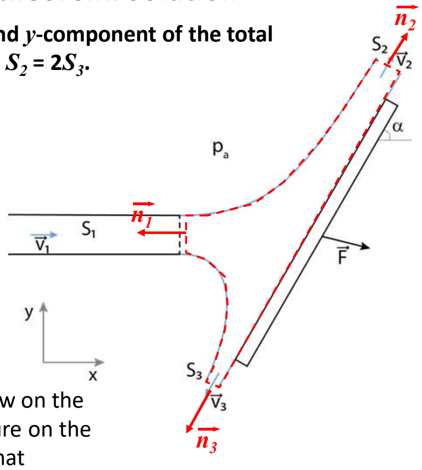
Ex 3. Euler momentum theorem. Solution

2. Choose a control volume and calculate the x- and y-component of the total force F the jet exerts on the plate. Assume that $S_2 = 2S_3$.



- Choose an appropriate control volume (in red in the Figure) in the jet bounded by surfaces Sa, S2 and S3 perpendicular to the jet in regions where the jet is quasi-straight.
- Apply the Euler momentum theorem:

$$\oint_S \rho\vec{V}\left(\vec{V}.\vec{n}\right)dS \quad = \quad \vec{F}_{p^*}$$

- $\vec{F}_{p^*} = -\vec{F}$ where F is the force exerted by the flow on the plate. F is obtained as the integral of the pressure on the surface of the control volume. Only pressures that deviate from the atmospheric pressure contribute to F. These deviating pressures occur on the plate.
- Assume that the velocity is constant in the cross-section of the jet: $\left|\vec{V}_1\right| = \left|\vec{V}_2\right| = \left|\vec{V}_3\right| = U$

$$\rightarrow \rho U^2 S_1\vec{n}_1 + \rho U^2 S_2\vec{n}_2 + \rho U^2 S_3\vec{n}_3 = -\vec{F}$$

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Ex 3. Euler momentum theorem. Solution

- Mass conservation: $Q_1 = Q_2 + Q_3 \rightarrow US_1 = US_2 + US_3 \rightarrow S_1 = S_2 + S_3$
- Input data: $S_2 = 2S_3$

$$\left.\begin{array}{l} S_1 = S_2 + S_3 \\ S_2 = 2S_3 \end{array}\right\} \rightarrow \begin{array}{l} S_2 = 2S_1/3 \\ S_3 = S_1/3 \end{array}$$

$$\rightarrow \rho U^2 S_1\left(\vec{n}_1 + \frac{2}{3}\vec{n}_2 + \frac{1}{3}\vec{n}_3\right) = -\vec{F}$$

- $$\left.\begin{array}{l} \vec{n}_1 = (-1,0) \\ \vec{n}_2 = (\cos 60°, \sin 60°) \\ \vec{n}_3 = (-\cos 60°, -\sin 60°) \end{array}\right\} \rightarrow \begin{cases} \rho U^2 S_1\left(-1 + \frac{1}{3} - \frac{1}{6}\right) = -F_x \\ \rho U^2 S_1\left(0 + \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{6}\right) = -F_y \end{cases} \rightarrow \begin{cases} F_x = 16.7 \text{ [kN]} \\ F_y = -5.8 \text{ [kN]} \end{cases}$$

Remark:
For an angle $\alpha = 0°$, this approach would lead to the result that the jet does not exert any force on the plate. This result is obtained because we have neglected friction between the plate and the jet.

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Ex 1. Bernoulli's equation

We have discussed in TH_Introduction the very original and surprising hunting strategy of the pistol shrimp. By quickly closing its clamp, the pistol shrimp generates a high velocity jet, which leads to the generation of a cavitation bubble. The pressure shock generated by the subsequent explosion of the cavitation bubble kills the pray.

Imagine that the pistol shrimp is at the bottom of a 0.5 [m] deep aquarium filled with water at 20 [°]. Estimate the velocity of the jet that the pistol shrimp creates.

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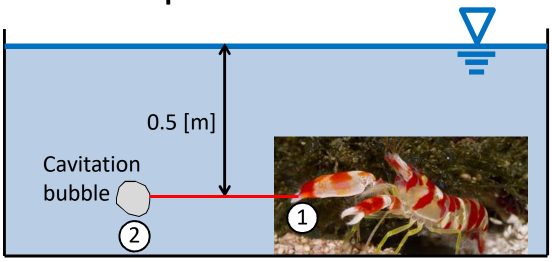
Ex 1. Bernoulli's equation



- Approximate the problem as a steady state flow
- Apply Bernoulli's equation to the streamline that originates at the pistol shrimp's claw (point 1) and terminates in the cavitation bubble (point 2):

$$h_1 + \frac{p_1}{\gamma} + \frac{U_1^2}{2g} = h_2 + \frac{p_2}{\gamma} + \frac{U_2^2}{2g}$$

- Assume $h_1 = h_2$ and $U_1 = 0$
- $p_1/\gamma = p_a/\gamma +$ 0.5 [m] = 101'360/9'810+0.5 [m] = 10.83 [m]. It is important to consider absolute pressure (including the atmospheric pressure) when dealing with cavitation.
- $p_2 = p_v$ = 2'339 [N m^{-2}] vapour pressure of water at 20 [°] (see TH_Introduction), i.e. the pressure at which water boils (and thus generates cavitation bubbles) at 20 [°].

$$\rightarrow U_2 = \sqrt{2g\left(\frac{p_1}{\gamma} - \frac{p_v}{\gamma}\right)} \approx 14 \text{ [m s}^{-1}\text{]}$$

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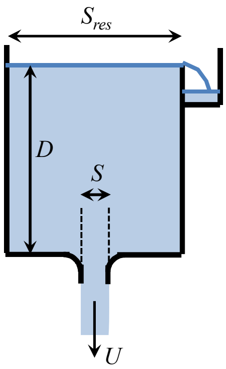
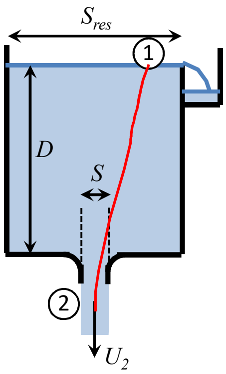
Ex 2. Bernoulli's equation. Torricelli's formula (1644)



Consider a reservoir of constant surface area S_{res} filled with a fluid of density ρ [kg m^{-3}]. An overflow is used to maintain the fluid at a constant level D [m] in the reservoir. The fluid flows out of the reservoir in the form of a jet through an opening in the bottom of surface area S [m^2].

1. Determine the velocity U [m s^{-1}] in the jet.
2. Determine the discharge flowing out through the bottom opening.
3. Consider now the emptying of the reservoir. Assume that $S << S_{res}$, such that the variation of the free surface in the reservoir is very slow, and the problem can be considered as quasi-stead. Determine the time required to empty the reservoir.

Ex 2. Bernoulli's equation. Torricelli's formula (1644). Solution



1. **Determine the velocity U [m s^{-1}] in the jet.**

- Apply Bernoulli's equation to the streamline that originates at the water surface (point 1) and terminates the free jet just below the opening (point 2):

$$h_1 + \frac{p_1}{\gamma} + \frac{U_1^2}{2g} = h_2 + \frac{p_2}{\gamma} + \frac{U_2^2}{2g}$$

- The pressure at the jet-air interface is atmospheric. In sections with quasi-straight and quasi-linear streamlines, the pressure does not vary perpendicularly to the streamline (TH_BernoulliEquation). As a result, $p = p_a$ in the jet just below the opening

- Mass conservation: $U_1 S_{res} = U_2 S$ if we assume constant velocities at the reservoir surface and in the jet.
- Because $S << S_{res}$, the variation of the free surface in the reservoir is very slow → $U_1 \approx 0$.

$\rightarrow U_2 = \sqrt{2gD}$: Torricelli's formula

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Ex 2. Bernoulli's equation. Torricelli's formula (1644). Solution

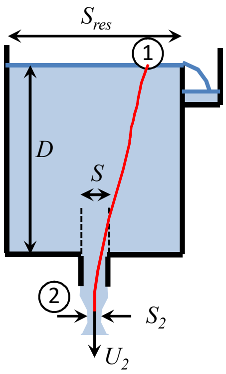
Note that the same results is obtained for an opening in the sidewall of the reservoir, as long as the size of the opening is small with respect to D.

There are two surprising things about Torricelli's formula:

1) The speed of efflux of a liquid from an opening in a reservoir equals the speed that the liquid would acquire if allowed to fall from rest from the surface of the reservoir to the opening. The velocity of a solid particle dropped in vacuum from a height D would be the same.
2) This speed of efflux is independent of the density of the fluid, i.e. oil and water have the same speed of efflux.

Explanation: By applying the Bernoulli equation along a streamline, we have neglected energy losses, i.e. we have assumed that the fluid has no resistance against deformation or flow, or in other words no viscosity.

Ex 2. Bernoulli's equation. Torricelli's formula (1644). Solution



2. **Determine the discharge flowing out through the bottom opening.**

$$Q = S_2 U_2 = \boxed{C_c} S U_2 = \boxed{C_c} S \sqrt{2gD}$$

C_c is the contraction coefficient.

- A contraction of the cross-section of the jet typically forms just downstream of the opening. The smallest cross-section is called the *vena contracta*. It cross-sectional area is defined as $S_2 = C_c S$, where S is the cross-sectional area of the orifice.
- The contraction coefficient C_c depends on the geometry of the opening. For a rounded opening (as designed in the previous figure), the contraction is negligible and $C_c \approx 1$. For sharp-edged openings, C_c can be as low as 0.6.

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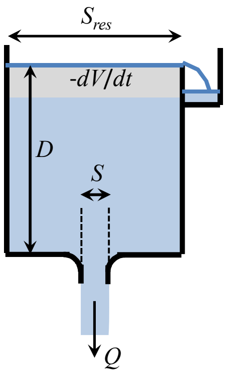
Ex 2. Bernoulli's equation. Torricelli's formula (1644). Solution



3. Determine the time t_e required to empty the reservoir.

According to mass conservation, the outflowing discharge Q leads to loss of fluid volume in the reservoir of -dV per unit time dt, or: Q=-dV/dt. This loss of fluid volume occurs as a lowering of the water surface in the reservoir, or $dV = S_{res}(D)dD$.

$$\rightarrow Q = C_c S\sqrt{2gD} = -S_{res}(D)\frac{dD}{dt}$$

$$\rightarrow dt = -\frac{1}{C_c S\sqrt{2g}}\frac{S_{res}(D)}{\sqrt{D}}dD$$

$$\rightarrow \int_0^{t_e} dt = t_e = -\frac{1}{C_c S\sqrt{2g}}\int_{D_0}^{0}\frac{S_{res}(D)}{\sqrt{D}}dD$$

;D_0 = initial water level in the reservoir

- Note that this formula is valid for an arbitrary shape of the reservoir.
- For a reservoir of constant cross-sectional shape (S_{res} = constant), the formula gives:

$$t_e = -\frac{S_{res}}{C_c S\sqrt{2g}}\int_{D_0}^{0}\frac{dD}{\sqrt{D}} = \frac{2S_{res}\sqrt{D_0}}{C_c S\sqrt{2g}}$$

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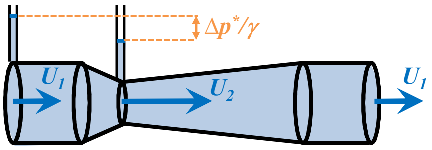
Ex 3. Bernoulli's equation. The Venturi tube (~1800)

A Venturi tube is a pipe that consists of a contraction followed by an expansion (Figure). Demonstrate that velocity and discharge can be derived from the pressure difference between the upstream section and the contracted section.

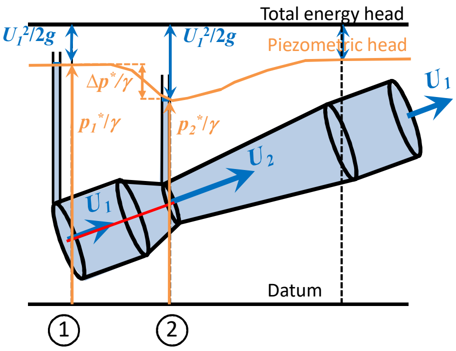
Note that energy losses are typically negligible in converging flow, whereas they can be substantial in diverging flows, especially when flow separates from the walls and recirculation zones form. For that reason, the contraction in Venturi tubes is typically rather abrupt (~20[°]), whereas the divergence is typically more gradual (~6[°]).

Venturi tube



Venturi tube on the hull of an airplane for velocity measurements

Ex 3. Bernoulli’s equation. The Venturi tube (~1800). Solution



Assume that the velocity is constant in the approach cross-section 1 and in the contracted cross-section 2. Assume further that streamlines are locally about straight and parallel, implying that p^* is constant in the cross-section (see TH_BernoulliEquation). This implies that the energy per unit head $E = p^*+U^2/2g$ is constant in the cross-section

Apply now Bernoulli’s equation on a streamline a a streamline that between a point in cross-section 1 and a point in cross-section 2:

$$\left.\begin{array}{l} \dfrac{p_1^*}{\gamma}+\dfrac{U_1^2}{2g}=\dfrac{p_2^*}{\gamma}+\dfrac{U_2^2}{2g} \\ \text{Mass conservation} \\ Q=U_1S_1=U_2S_2 \end{array}\right\} \rightarrow \frac{\Delta p^*}{\gamma}=\frac{p_1^*}{\gamma}-\frac{p_2^*}{\gamma}=\frac{Q^2}{2g}\left(\frac{1}{S_2^2}-\frac{1}{S_1^2}\right) \rightarrow \boxed{Q=C_Q U_2 S_2=\frac{\sqrt{2g\dfrac{\Delta p^*}{\gamma}}}{\sqrt{1-\left(\dfrac{S_2}{S_1}\right)^2}}S_2}$$

The empirical discharge coefficient C_Q is often introduced to account for deviations from theory, such as the occurrence of (relatively small) energy losses in the Venturi. 24

Ex 1. Open-channel flow

Consider a reach on the Danube near Vienna.

1. Schematize the geometry of the river system and justify your schematization.

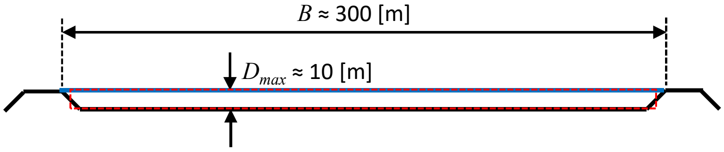
   In practice, it is important to treat problems with the appropriate level of complexity. For example, what level of complexity do you retain in the schematization of the river shape ? Can the cross-sectional shape be approximated by a trapezium ? Or is the effect of the banks negligible and can it be approximated by a rectangle, which simplifies calculations. Can the bottom slope be taken as constant in the considered reach ?

2. Choose a discharge Q (for example the mean annual discharge).
3. Draw the specific energy curves for Q.
4. Compute the critical flow depth (D_c) and the corresponding specific energy ($E_{s,c}$) for this Q and indicate them on the specific energy curve.
5. Make an estimation of the friction coefficient and justify your estimation.
6. Compute the normal flow depth (D_n) for Q, and represent it on the specific energy curve.
7. Define the flow regime.
8. Due to construction works, the width has to be reduced by 50m over a length of 500m. Based on specific energy considerations, compute the local variation in the elevation of the water surface resulting from this width reduction.

Ex 1. Open-channel flow. Solution

1. Schematize the geometry of the river system and justify your schematization.

- We can measure on a map or Google Earth, for example, that the top width is about 300 [m].
- We can measure the bed slope on a map, Google Earth, or search in literature. The longitudinal profile of the Danube has been illustrated in TH_Introduction. That figure shows a slope that is quasi constant near Vienna, and approximately equal to J_f = 0.0005 [-].
- The cross-sectional shape can be approximated by a trapezium with horizontal bottom and inclined banks. The maximum flow depth near Vienna is about 10 [m]. This implies a minimum width-to-depth ratio of B/D_{max} = 30. It is generally accepted that the inclination of the banks can be neglected for channels with B/D > 10. This implies that the cross-sectional shape can be approximated by a rectangular. Note that the computations for a rectangular cross-section are simpler than for a trapezoidal one. The trapezoidal and approximate rectangular (red line) cross-sections are drawn on scale in the figure below, which shows convincingly that the effects of the bank inclination can be neglected.



This level of approximation is sufficient for most practical engineering applications. Note also, that more detailed geometrical input data are often not available in engineering applications.

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Ex 1. Open-channel flow. Solution

2. Choose a discharge Q (for example the mean annual discharge).

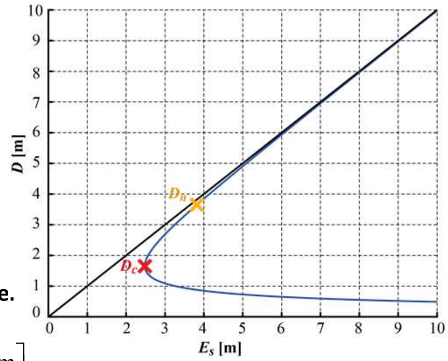
The annual average discharge near Vienna is $Q \approx 2000$ [m^3 s^{-1}]. This result can be obtained from hydrological data measured by the Austrian hydrological services. It can also be read from the figure shown in TH_Introduction.

Notice that the methodology in Ex 1 and Ex 2 can also be adopted to analyze another discharge, such as a flood discharge.

3. Draw the specific energy curves for Q.

$$E_s = D + \frac{U^2}{2g} = D + \frac{Q^2}{2g(BD)^2}$$

The curve has been computed with Matlab for D in the range 0 to D_{max} = 10 [m]



4. Compute the critical flow depth (D_c) and the corresponding specific energy ($E_{s,c}$) for this Q and indicate them on the specific energy curve.

$$\frac{Q^2 B}{gS^3} = \frac{Q^2}{gB^2 D_c^3} = 1 \quad \rightarrow D_c = \left[\frac{(Q/B)^2}{g}\right]^{\frac{1}{3}} = 1.65\,[\mathrm{m}]$$

and $E_{s,c}$ = 2.48 [m]

Ex 1. Open-channel flow. Solution

5. Make an estimation of the friction coefficient and justify your estimation.

The estimation of the friction coefficient is of crucial importance but very difficult, as discussed in TH_OpenChannelFlow1. The Danube in the Vienna reach is obviously intensively monitored by the Austrian hydrological services. Hydrological stations operate that measure continuously the flow depth (derived from the water surface elevation) and occasionally the discharge (by integration of velocities measured on a grid of points spanning the cross-section). The friction coefficient can be estimated from the measured Q-D curve under the assumption of normal flow.

Note that numerous publications deal with the Danube reach near Vienna. These publications are a reliable source for the estimation of the friction factor.

For the considered Danube reach, a Manning-Strickler friction coefficient of K_s = 35 [m$^{1/3}$ s^{-1}] is adopted here.

6. Compute the normal flow depth (D_n) for Q, and represent it on the specific energy curve.

$$Q = US = K_s R_h^{2/3} J_f^{1/2} S = K_s D_n^{2/3} J_f^{1/2} B D_n = K_s J_f^{1/2} B D_n^{5/3} \quad \rightarrow D_n = \left(\frac{Q}{K_s J_f^{1/2} B} \right)^{\frac{3}{5}} = 3.65 \left[\mathrm{m} \right]$$

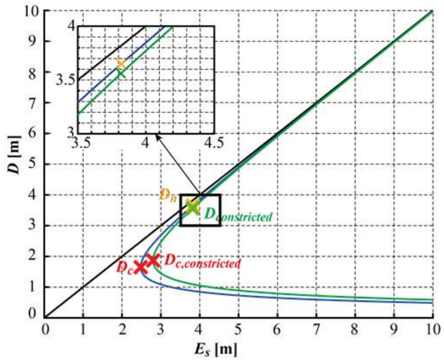
7. Define the flow regime.

For the considered discharge, $D_n > D_c$, which implies that the normal flow at this discharge is a subcritical flow with $Fr < 1$. It also implies that this Danube reach is a mild-slope channel.

Ex 1. Open-channel flow. Solution

8. Due to construction works, the width has to be reduced by 50m over a length of 500m. Based on specific energy considerations, compute the local variation in the elevation of the water surface resulting from this width reduction.

- We first establish the specific energy curve for this constricted reach (green curve in the figure), and superpose it on the curve for the unconstricted reach.
- The specific energy for the normal flow, $E_{s,n}$ = 3.82 [m], is larger than the minimum required specific energy in the constricted reach, $E_{s,c,constricted}$ = 2.80 [m]. This means that the constriction does not perturb the flow in the unconstricted upstream reach.
- The flow depth in the constricted reach is found by expressing that $E_{s,constricted} = E_{s,n}$ = 3.82 [m] and by subsequently deriving the corresponding flow depth $D_{constricted}$ = 3.56 [m] from the specific energy curve (see insert in the figure). The constriction causes a drop in the water surface elevation of 0.09 [m].



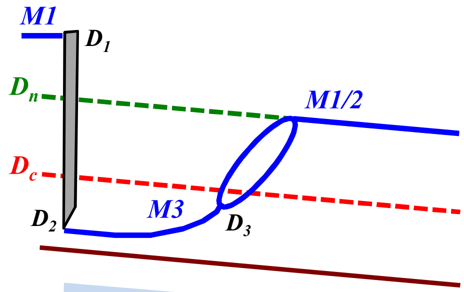
Ex 2. Open-channel flow

Consider the same reach on the Danube as in Ex. 1 and the same discharge. Assume that the reach has a constant geometry and is sufficiently long for normal flow conditions to establish. A sluice gate is installed over the entire width that locally reduces the flow depth to 0.5 [m].

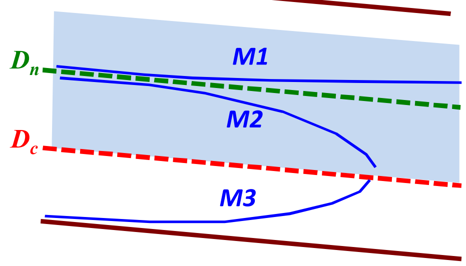
1. Draw schematically the backwater curves upstream and downstream of the sluice gate. Indicate in your scheme the normal and critical flow depths, and name the types of backwater curve that occur.
2. Compute the backwater curves upstream and downstream of the sluice gate.
3. If a hydraulic jump occurs, determine the conjugate depths and determine its location (distance from the sluice gate). The length of the hydraulic jump can be neglected.

Ex 2. Open-channel flow. Solution

1. **Draw schematically the backwater curves upstream and downstream of the sluice gate. Indicate in your scheme the normal and critical flow depths, and name the types of backwater curve that occur.**



- We are dealing with the mild-slope channel case considered and schematically drawn in TH_OpenChannelFlow3. The figure from TH_OpenChannelFlow3 is reproduced beside.



- In a mild-slope channel of constant geometry (bottom slope and cross-sectional shape), the flow depth tends to the normal flow depth in upstream direction. Once the normal flow depth has been attained, it cannot change anymore (only M1 and M2 backwater curves are possible).

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Ex 2. Open-channel flow. Solution

- The supercritical flow depth at the sluice gate (D_2 = 0.5 [m] < D_c = 1.65 [m]) can only connect with the subcritical normal flow further downstream by means of a hydraulic jump. The hydraulic jump occurs between a to-be-determined supercritical flow depth D_3 at the upstream end of the hydraulic jump and the conjugate normal flow depth D_n at the downstream end of the hydraulic jump.
- D_3 can be determined from the equation (TH_OpenChannelFlow3):

$$\frac{D_3}{D_n} = \frac{1}{2}\left(\sqrt{1+8Fr_n^2} - 1\right) \quad ; D_n = 3.65 \text{ [m] and } Fr_n = \sqrt{\frac{Q^2 B}{gS^3}} = \sqrt{\frac{Q^2}{gB^2 D_n^3}} = 0.305 \rightarrow D_3 = 0.59 \text{ [m]}$$

- D_3 = 0.59 [m] > D_2 = 0.5 [m], implying that an M3 backwater curve will develop between D_2 and D_3. Note that for the case D_3 < D_2 < D_c, a so-called submerged hydraulic jump develops (which is beyond the scope of the present course).
- The water level upstream of the sluice gate will rise, in order to generate the pressure gradient required to make the entire discharge pass under the sluice gate. The water level D1 upstream of the sluice gate can be estimated by expressing that energy is conserved in the converging flow under the sluice gate:

$$E_{s,1} = D_1 + \frac{Q^2}{2g\left(BD_1\right)^2} = E_{s,2} = D_2 + \frac{Q^2}{2g\left(BD_2\right)^2} = 9.56 \text{ [m]} \rightarrow D_1 = 9.56 \text{ [m]}$$

- An M1 backwater curve will connect D_1 to the normal flow depth D_n further upstream.

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Ex 2. Open-channel flow. Solution

2. Compute the backwater curves upstream and downstream of the sluice gate.
3. If a hydraulic jump occurs, determine the conjugate depths and determine its location (distance from the sluice gate). The length of the hydraulic jump can be neglected.

- As mentioned in the lecture, it is good practice to start computing only when you know the solution already qualitatively. So we are ready now to start computing.
- We have seen in TH_OpenChannelFlow2 that the backwater curve can be computed by discretizing the derivative *dD/dx:*

$$\frac{dD}{dx} = \frac{J_f - J_e}{1 - Fr^2} = fct(D) \rightarrow \frac{D_{i+1} - D_i}{x_{i+1} - x_i} = fct(D_{i+1/2}) \approx 0.5\left[fct(D_i) + fct(D_{i+1}) \right]$$

Most often, the longitudinal x axis is discretized into a regular grid with spacing Δx constant, and the flow dept D_i is computed in every grid point x_i following an iterative procedure (see TH_OpenChannelFlow2). Iteration is required, because the unknown D_{i+1} appears in the left hand side and the right hand side of the equation.
But there is a more clever way of solving the equation in the case considered. We can also discretize the D range in regular ΔD intervals, and compute the location x_i where a flow depth D_i occurs. This procedure is appropriate in the considered case, because we know all relevant flow depths from upstream to downstream: $D_n \rightarrow D_1 \rightarrow D_2 \rightarrow D_3 \rightarrow D_n$. Solving the resulting equation does not require iteration:

$$x_{i+1} - x_i = \frac{D_{i+1} - D_i}{0.5\left[fct(D_i) + fct(D_{i+1}) \right]}$$

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Ex 2. Open-channel flow. Solution

The equation can be written more explicitly as:

$$fcd(D)=\frac{J_f-J_e}{1-Fr^2}=J_f\frac{1-\dfrac{J_e}{J_f}}{1-\dfrac{Fr^2}{1}}=J_f\frac{1-\dfrac{\dfrac{Q^2}{K_s^2R_h^{4/3}S^2}}{\dfrac{Q^2}{K_s^2R_{h,n}^{4/3}S_n^2}}}{1-\dfrac{\dfrac{Q^2}{gD_hS^2}}{\dfrac{Q^2}{gD_{h,c}S_n^2}}}=J_f\frac{1-\dfrac{R_{h,n}^{4/3}S_n^2}{R_h^{4/3}S^2}}{1-\dfrac{D_{h,c}S_n^2}{D_hS^2}}$$

- In TH_OpenChannelFlow2, we have derived a similar equation based on the Chézy friction coefficient. Note that the exponent of R_h is 4/3 when modelling the energy slope with the Manning-Strickler friction coefficient, and 1 when using the Chézy friction coefficient. For a wide rectangular channel, this equation simplifies into:

$$fcd(D)=J_f\frac{1-\left(\dfrac{D_n}{D}\right)^{10/3}}{1-\left(\dfrac{D_c}{D}\right)^{3}}$$

Ex 2. Open-channel flow. Solution

This leads to the following equation for the backwater curves:

$$x_{i+1} - x_i = \frac{D_{i+1} - D_i}{0.5 J_f \left[\dfrac{1-\left(\dfrac{D_n}{D_i}\right)^{10/3}}{1-\left(\dfrac{D_c}{D_i}\right)^{3}} + \dfrac{1-\left(\dfrac{D_n}{D_{i+1}}\right)^{10/3}}{1-\left(\dfrac{D_c}{D_{i+1}}\right)^{3}} \right]}$$

Downstream of the sluice gate:

- The backwater curve between D_2 = 0.5 [m] and D_3 = 0.59 [m] is very short and can be computed in one step. Solving the equation, we find that the hydraulic jump occurs 8.5 [m] downstream of the sluice gate. In practice, this means immediately downstream of the sluice gate.
- Note that the supercritical velocities downstream of the sluice gate are very high and have an enormous erosion potential. In practice, the bottom has to be reinforced in the supercritical flow reach. In order to make this supercritical flow reach as short as possible and to fix the location of the hydraulic jump, a stilling basin is designed just downstream of the sluice gate. Stilling basins will be considered in the course "Konstruktiver Wasserbau".

Ex 2. Open-channel flow. Solution

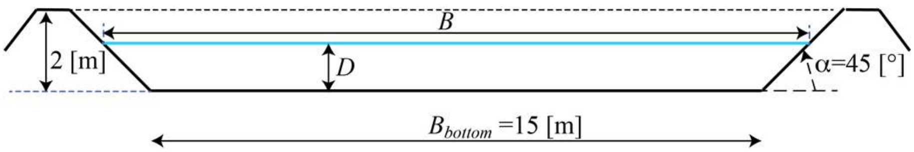
Upstream of the sluice gate:

- Here we compute from the sluice gate where D_1 = 9.56 [m] and evolve in upstream direction towards D_n = 3.65 [m].
- The water surface gradient dD/dx is small for an M1 curve. This allows choosing rather large steps $D_{i+1} - D_i$.

D_i [m]	9.56	8.65	7.65	6.65	5.65	4.65	3.65
Δx [m]		-1901	-2134	-2219	-2403	-2928	-6899
x [m]	0	-1901	-4035	-6254	-8657	-11585	-18484

Ex 3. Open-channel flow

Consider a mountain river with the following characteristics:
- A longitudinal bottom slope of J_f = 0.02.
- A trapezoidal cross-section with bottom width of B_{bottom} = 15 [m], banks inclined at α = 45[°], and bank height of 2 [m] (Figure).
- A roughness coefficient according to Manning-Strickler of K_s = 30 [m$^{1/3}$ s^{-1}].



1. Compute the hydraulic capacity of the river, which is also called the bankfull discharge. Assume that flow is normal.
2. Compute the normal and critical flow depths for the bankfull discharge.
3. Identify the flow regime.
4. Draw the specific energy curve for the bankfull discharge and indicate the normal and critical flows. Consider a depth range of 0 to 6 m for drawing the curve.
5. Compute the bed shear stress for the bankfull discharge.

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Ex 3. Open-channel flow. Solution

1. Compute the hydraulic capacity of the river, which is also called the bankfull discharge. Assume that flow is normal.

The discharge for normal flow conditions is given by (see TH_OpenChannelFlow1):

$$Q = US = K_s R_h^{2/3} J_f^{1/2} S = K_s \left(\frac{S}{P}\right)^{2/3} J_f^{1/2} S = K_s \frac{S^{5/3}}{P^{2/3}} J_f^{1/2}$$

where $R_h = S/P$ is the hydraulic radius, S the cross-sectional flow area, and P the wetted perimeter, under normal flow conditions. For a trapezoidal cross-section S and P are defined as:

$$\left.\begin{aligned} S &= \left(B_{bottom} + \frac{D}{\tan\alpha}\right) D \\ P &= B_{bottom} + \frac{2D}{\sin\alpha} \end{aligned}\right] \rightarrow Q = K_s \frac{\left[\left(B_{bottom} + \frac{D}{\tan\alpha}\right) D\right]^{\frac{5}{3}}}{\left[B_{bottom} + \frac{2D}{\sin\alpha}\right]^{\frac{2}{3}}} J_f^{1/2}$$

→ $Q_{bankfull}$ = 201 [m^3 s^{-1}] for D = 2 [m]

2. Compute the normal and critical flow depths for the bankfull discharge.

The normal flow depth is D_n = 2 [m] as found in the previous question.

The critical flow depth is found from the equation $\frac{Q^2 B}{gS^3} = 1$ (TH_OpenChannelFlow1)

where $Q = Q_{bankfull}$ and B is the top width (i.e. the width at the water surface):

$$B = B_{bottom} + \frac{2D}{\tan\alpha}$$

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Ex 3. Open-channel flow. Solution

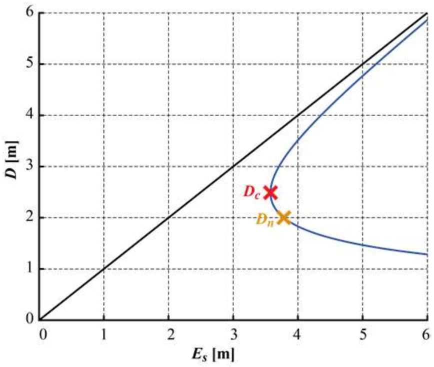
$$\rightarrow Q^2_{bankfull}\frac{\left(B_{bottom}+\frac{2D_c}{\tan\alpha}\right)}{\left[g\left(B_{bottom}+\frac{D_c}{\tan\alpha}\right)D_c\right]^3}-1=0$$ Solving this implicit equation leads to D_c = 2.49 [m]

3. Identify the flow regime

For bankfull discharge, $D_n < D_c$, which implies that the normal flow at bankfull discharge is a supercritical flow with $Fr > 1$. It also implies that we are dealing with a steep-slope channel.

4. Draw the specific energy curve for the bankfull discharge and indicate the normal and critical flows. Consider a depth range of 0 to 6 m for drawing the curve.

$$E_s = D+\frac{U^2}{2g}=D+\frac{Q^2}{2g\left[\left(B_{bottom}+\frac{D}{\tan\alpha}\right)D\right]^2}$$



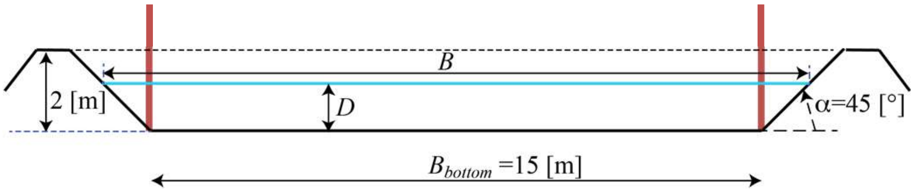
5. Compute the bed shear stress for the bankfull discharge

$$\tau_b=\rho g D J_e \overset{normal\ flow}{=} \rho g D_n J_f = 392.4\left[\frac{\mathrm{N}}{\mathrm{m}^2}\right]$$

(see TH_OpenChannelFlow1)

Ex 4. Open-channel flow

Consider the same mountain river as in Ex. 3. Due to the crossing of a highway bridge, the banks are locally vertical, but the bottom width is maintained at 15 m, leading to a local constriction of the flow (Figure).



1. Draw the specific energy curve in the constricted reach for the bankfull discharge identified in Ex. 3. Superpose this on the specific energy curve drawn in Ex. 3.
2. By how much do the banks have to be raised in order to maintain the hydraulic capacity, i.e. in order to avoid inundations.
3. A hydraulic jump will occur upstream of the constriction. Compute the conjugate flow depths (i.e. flow depths just upstream and downstream of the hydraulic jump) and compute the energy losses in the hydraulic jump.
4. Draw schematically the longitudinal profiles of the bed, water surface and energy line in the reach upstream of the constriction; indicate also the normal and critical flow depths.
5. Illustrate the evolution of the water depth on the specific energy curve.

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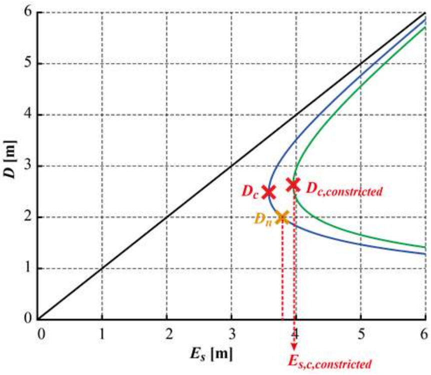
Ex 4. Open-channel flow. Solution

1. Draw the specific energy curve in the constricted reach for the bankfull discharge identified in Ex. 3. Superpose this on the specific energy curve drawn in Ex. 3.

$$E_{s,constricted} = D + \frac{U^2}{2g} = D + \frac{Q^2}{2g\left(B_{bottom}D\right)^2}$$



The specific energy curve in the constricted reach for $Q_{bankfull}$ is drawn in green.

2. By how much do the banks have to be raised in order to maintain the hydraulic capacity, i.e. in order to avoid inundations

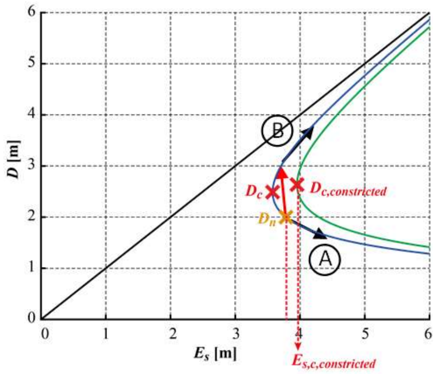
In order to answer this question, we have to analyze the effect of the constriction on the water surface elevation.

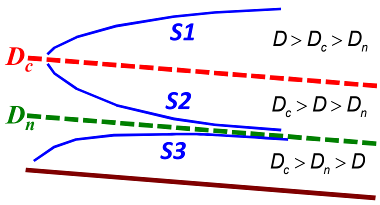
The specific energy at normal flow in the upstream reach $E_{s,n}$ is smaller than the specific energy for critical flow in the constricted reach $E_{s,c,constricted} = 3.95$ [m], which is the minimum specific energy to convey the bankfull discharge through the constricted reach. In other words, the normal flow upstream has not enough specific energy to pass the constriction. The flow depth upstream will have to change in such as way that the available specific energy increases.

Ex 4. Open-channel flow. Solution

There are two possible ways to increase the specific energy:

1. The flow depth can decrease in supercritical flow regime (scenario A in the specific energy curve).
   But there is no possible backwater curve that allows the normal flow depth to decrease in downstream direction in a steep-slope channel (see figure). Once the normal flow depth has been attained in a reach of constant geometry (bed slope and cross-sectional shape), it cannot change anymore.
2. The flow depth can increase in subcritical flow regime (scenario B in the specific energy curve).
   This requires first a change from supercritical to subcritical flow regime, which occurs through a hydraulic jump (red arrow on specific energy curve). The S1 backwater curve than allows for increasing specific energy accompanied by increasing flow depth.





→ We first have to answer questions 3 before we can answer question 2.

Ex 4. Open-channel flow. Solution

3. A hydraulic jump will occur upstream of the constriction. Compute the conjugate flow depths (i.e. flow depths just upstream and downstream of the hydraulic jump) and compute the energy losses in the hydraulic jump.

The hydraulic jump will occur between normal supercritical flow depth upstream, and its conjugate subcritical flow downstream, which is found from (TH_OpenChannelFlow3):

$$\frac{D_2}{D_n}=\frac{1}{2}\left(\sqrt{1+8Fr_n^2}-1\right)$$ where D_n = 2 [m] and $$Fr_n=\sqrt{\frac{Q^2B}{gS^3}}=\sqrt{\frac{Q^2\left(B_{bottom}+\frac{2D_n}{\tan\alpha}\right)}{g\left[\left(B_{bottom}+\frac{D_n}{\tan\alpha}\right)D_n\right]^3}}=1.41$$

→ D_2 = 3.12 [m]

The energy losses in the hydraulic jump are computed from (TH_OpenChannelFlow3):

$$\Delta E=\Delta E_s=\left(D_1+\frac{U_1^2}{2g}\right)-\left(D_2+\frac{U_2^2}{2g}\right)=\frac{\left(D_2-D_1\right)^3}{4D_1D_2}$$ = 0.056 [m]

With this information, we can come back to question 2.

Ex 4. Open-channel flow. Solution

2. By how much do the banks have to be raised in order to maintain the hydraulic capacity, i.e. in order to avoid inundations

The specific energy corresponding to D_2 is: $E_{s,2} = D_2 + \dfrac{Q^2}{2g\left[\left(B_{bottom} + \dfrac{D_2}{\tan\alpha}\right)D_2\right]^2} = 3.76$ [m]

$E_{s,2}$ = 3.76 [m] < $E_{s,c,constricted}$ = 3.95 [m]. This means that the specific energy just downstream of the hydraulic jump is still not sufficient. As discussed before, a further increase in specific energy occurs through an increase in flow depth according to an S1 backwater curve. The flow depth will rise until a flow depth D_3, that has just enough specific energy, i.e. until $E_{s,3}$ = 3.95 [m].

$$E_{s,3} = 3.95 \text{ [m]} = D_3 + \frac{Q^2}{2g\left[\left(B_{bottom} + \dfrac{D_3}{\tan\alpha}\right)D_3\right]^2} \rightarrow D_3 = 3.44 \text{ [m]}$$

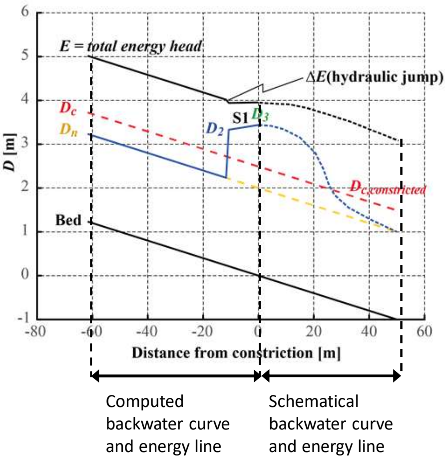
At this flow depth D3, the flow will change from subcritical to supercritical regime through a sudden water level drop.

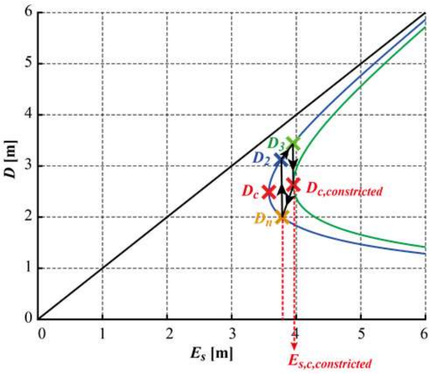
The maximum flow depth is D_{max} = D_3 = 3.44 [m] implying that banks have to be raised by 1.44 [m] to avoid inundations. Note that the maximum flow depth occurs upstream of the constricted reach !

Ex 4. Open-channel flow. Solution

4. Draw schematically the longitudinal profiles of the bed, water surface and energy line in the reach upstream of the constriction; indicate also the normal and critical flow depths.

5. Illustrate the evolution of the water depth on the specific energy curve.





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Ex 5. Open-channel flow

We have seen in TH_OpenChannelFlow_1 that discharge can be measured by imposing critical flow conditions. We have treated the example of imposing critical flow by means of a bottom step (the relevant slide is reproduced in the figure below). Develop explicitly the relation $Q = Q\,(D_{upstream})$ for a rectangular channel.

Discharge measurements based on relation *Q-D$_c$*

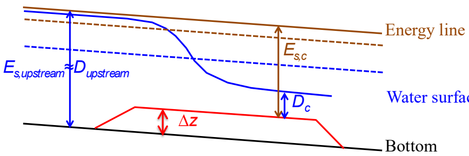
Critical flow:

$D_c = D_c(Q)$ and $Q = Q(D_c)$

$E_{s,c} = E_{s,c}(Q)$ and $Q = Q(E_{s,c})$

$E_{s,c} \approx E_{upstream} - \Delta z \approx D_{upstream} - \Delta z$

$\rightarrow Q = Q(D_{upstream})$



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Ex 5. Open-channel flow

Critical flow is defined by:

$$\frac{Q^2}{gS^3(D_c)}B(D_c)=1 \rightarrow Q=g^{1/2}S^{3/2}(D_c)B^{-1/2}(D_c) \longrightarrow Q=g^{1/2}BD_c^{3/2}$$

The specific energy for critical flow is:

$$E_{s,c}=D_c+\frac{U_c^2}{2g}=D_c+\frac{Q^2}{2gS^2(D_c)}=D_c+\frac{S(D_c)}{2B(D_c)} \longrightarrow E_{s,c}=\frac{3}{2}D_c$$

$$E_{s,c}\approx E_{upstream} - \Delta z \approx D_{upstream} - \Delta z$$

Rectangular channel: $S = BD$

$$Q=g^{1/2}BD_c^{3/2}$$

$$E_{s,c}=\frac{3}{2}D_c$$

$$Q=g^{1/2}B\left[\frac{2}{3}\left(D_{upstream}-\Delta z\right)\right]^{3/2}$$

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Ex 6. Open-channel flow

A discharge of Q = 12 [m^3 s^{-1}] is flowing in a 2 [m] wide rectangular channel. The Manning-Strickler roughness coefficient is estimated at K_s = 40 [$m^{1/3}$ s^{-1}]. The flow regime will obviously depend on the channel slope. For a mild slope the flow will tend to be subcritical, whereas it will tend to be supercritical for a steep slope. Determine the critical slope, i.e. the one that differentiates between mild-slope (*M*-type backwater curves) and steep-slope (S-type backwater curves) characterizations of the channel for the given discharge.

Ex 4. Open-channel flow. Solution

- Defintions (TH_OpenChannelFlow2): $D_n > D_c$ for a mild-slope channel, $D_n < D_c$ for a steep-slope channel, and $D_n = D_c$ at critical slope.
- D_c only depends on the discharge and the cross-sectional shape (TH_OpenChannelFlow1)
- D_n depends on the discharge, the cross-sectional shape, the friction coefficient, and the bottom slope (TH_OpenChannelFlow1)

→ For a given discharge, cross-sectional shape and friction coefficient, the channel characterization (mild-slope or steep-slope) will depend on the bottom slope.

- Let us first compute D_c from: $\frac{Q^2 B}{gS^3} = \frac{Q^2}{gB^2 D_c^3} = 1 \quad \rightarrow D_c = \left[\frac{(Q/B)^2}{g}\right]^{\frac{1}{3}} = 1.54\,[\mathrm{m}]$
- At critical slope, $D_n = D_c = 1.54$ [m]
- The relation between Q and D_n is given by (TH_OpenChannelFlow1):

$Q = US = K_s R_h^{2/3} J_f^{1/2} S$ with K_s the Manning-Strickler friction coefficient

$$\rightarrow J_{f,crit} = \left(\frac{Q}{K_s R_h^{2/3} S}\right)^2 = \left(\frac{Q}{K_s \left(\frac{BD_n}{B+2D_n}\right)^{2/3} (BD_n)}\right)^2 = 0.0184\,[-]$$

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