Fundamentals of Numerical Thermo-Fluid Dynamics 322.061
Examples for home preparation

Exercise 1: Classification of Partial Differential Equations

To be presented on May 15, 2019

1.1) Show that the time-dependent diffusion equation

$$\frac{\partial \phi}{\partial t} - \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0. \tag{1}$$

is parabolic. Transform it to an equivalent system of three first-order equations by introducing auxiliary unknowns $p = \frac{\partial \phi}{\partial x}$, $q = \frac{\partial \phi}{\partial y}$. Prove that the resulting system is also parabolic.

1.2) What is the type of equation (1) on coordinate planes $y = const.$ and $t = const.$?

1.3) Convert the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{2}$$

into an equivalent system of first-order equations by introducing auxiliary unknowns. Prove that the resulting system is parabolic and compute the characteristic.

1.4) One dimensional propagation of acoustic waves in fluids can be modelled as an adiabatically compressible inviscid flow by the following system of equations:

$$
\begin{aligned}
&\text{conservation of mass:} && \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \\
&\text{conservation of momentum:} && \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\
&\text{speed of sound:} && \frac{\partial p}{\partial \rho} = a^2
\end{aligned} \tag{3}
$$

where the unknowns u, ρ, p are fluid velocity, density and pressure respectively and the constant a is the speed of sound. Show that this system is hyperbolic and compute the characteristics. What can you conclude regarding propagation of pressure disturbance from certain point in space and time?

Hint: First eliminate pressure from the momentum equation, using the definition of the speed of sound. This would lead to a system of two first order equations with two unknowns u and ρ