Fundamentals of Numerical Thermo-Fluid Dynamics 322.061
Examples for home preparation

Exercise 3: Stability of Finite Difference schemes

To be presented on May 29, 2019

3.1) Let us consider a convection equation of passive scalar

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \tag{1}$$

which we discretize with second order central differences in space and Lax-Friedrichs method in time:

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\Delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O\left(\Delta t, \Delta x^2\right) \tag{2}$$

Investigate the stability of this scheme using von Neumann analysis. Provide the conditions of stability if relevant.

3.2) Consider a semi-discrete problem

$$\frac{\partial u_j}{\partial t} = f_j(\vec{u}) \tag{3}$$

Show that Lax-Friedrichs method

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\Delta t} = f_j(\vec{u^n}) \tag{4}$$

is equivalent to stabilizing the explicit Euler scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = f_j^n(\vec{u^n}) \tag{5}$$

by adding an artificial diffusion

$$\epsilon = \frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 u}{\partial x^2} + \begin{bmatrix} \text{high order} \\ \text{terms} \end{bmatrix} \tag{6}$$

Hint: Compute the error of the approximation $u_j^n \approx \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$

3.3) Implement the discretization (2) in MatLab in the form

$$\vec{u^{n+1}} = \mathbf{A}\vec{u^n} \tag{7}$$

Take $x_1 = 0, x_N = 1, \Delta x = 0.05, \Delta t = 0.02, c = 1$. Apply the boundary condition

$$u_1^{n+1} = u_1^n \quad \text{at} \quad x_1 = 0$$

and the one-sided difference

$$u_N^{n+1} = \frac{c\Delta t}{\Delta x}u_{N-1}^n + \left(1 - \frac{c\Delta t}{\Delta x}\right)u_N^n \quad \text{at} \quad x_N = 1$$

Assuming an initial condition

$$u(x, t_0 = 0) = e^{-\frac{(x-0.5)^2}{0.08}}, \tag{8}$$

compute the solution at $t = 0.2$. Then show that one can solve equation (1) with the method of characteristics to obtain the exact solution

$$u(x,t) = e^{-\frac{(x-ct-0.5)^2}{0.08}} \tag{9}$$

and compare your numerical result with this analytical solution. Explain your observations.

Hint: Consider question 3.2) for interpretation.

Bonus: Try to use different values of $\Delta x, \Delta t$ to verify the results of questions 3.1) and 3.2)

3.4) Discretize the equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\pi^2 \sin(\pi x)\cos(\pi y) \ , \ \ x, y \in [0, 1] \tag{10}$$

and write it in matrix and vector form (no programming required) using a second order central difference scheme for both the variables, considering the boundary conditions:

$$\begin{aligned} &(a) \ \ (x, 0) = \sin(\pi x), \\ &(b) \ \ (x, 1) = -\sin(\pi x), \\ &(c) \ \ (0, y) = 0, \\ &(d) \ \ (1, y) = 0, \end{aligned} \tag{11}$$

Use $\Delta x = 0.25$ and $\Delta y = 0.2$.