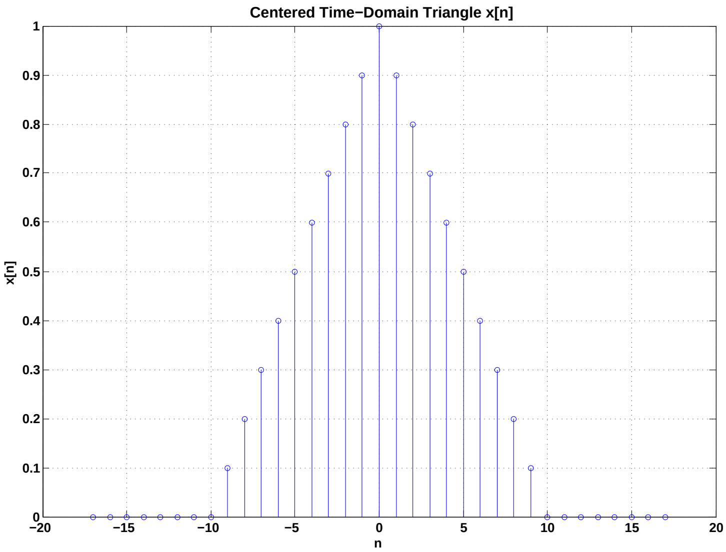
# 389.055: Signals and Systems 2
# Solution: A3.1 g)

Clemens Novak, Johannes Gonter
johannes.gonter@nt.tuwien.ac.at

March 25, 2010

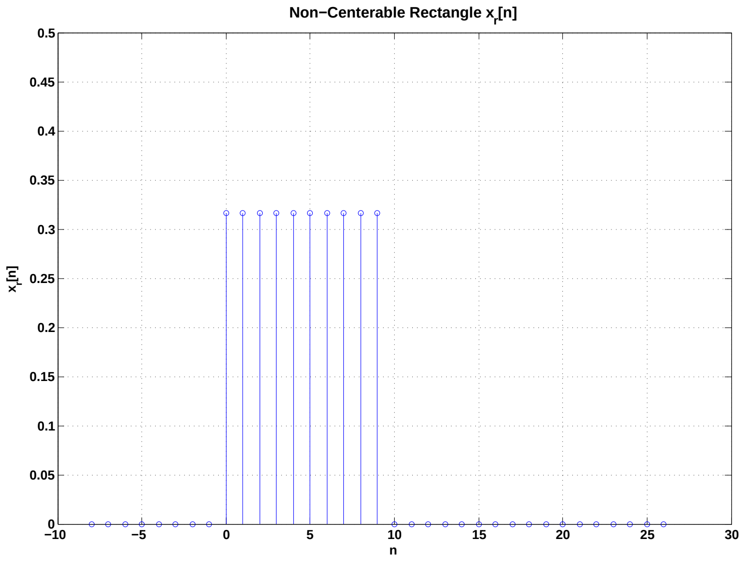
TASK: find the frequency-domain representation $X\left(e^{j\theta}\right)$ of the given discrete-time function $x[n]$:




HINT: Generate the triangle $x[n-9]$ by convolving the function $x_r[n]$ with itself and shift the result in the *frequency-domain* so that the triangle is centered, and:

$$x_r[n] = \frac{1}{\sqrt{10}}\left[\sigma[n] - \sigma[n-10]\right] \tag{1}$$

The triangle spans $-9 \ldots 9$, where it is important to note that 9 is an *odd* number. However, convolving a *symmetric* function with itself will always produce a function which starts and ends at *even* time-instances.




The triangle of overall length 19 requires the convolution of a rectangle of overall duration 10 with itself, which cannot be centered in time-domain, since there is *no central element*. However, the convolution can be calculated for a rectangle spanning $0...9$ and shifting the triangle in *frequency domain*:

$$x[n-9] = x_r[n] * x_r[n] \circ\!\!-\!\!\bullet\; X_r\left(\mathrm{e}^{j\theta}\right) \cdot X_r\left(\mathrm{e}^{j\theta}\right) \tag{2}$$

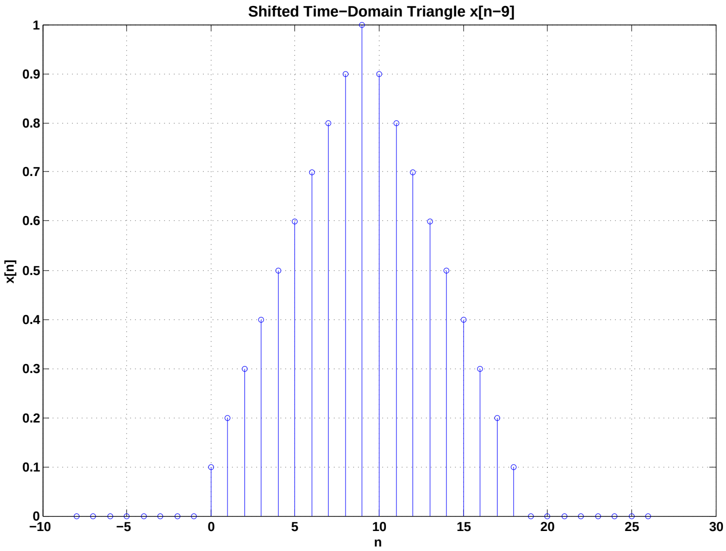
## I. Elegant Solution

$x_r[n]$ can be assembled of 10 weighted (with a factor of $\sqrt{(10)}$) *dirac pulses*, and its spectrum $X_r\left(\mathrm{e}^{j\theta}\right)$ is therefore given as:

$$X_r\left(\mathrm{e}^{j\theta}\right) = \frac{1}{\sqrt{(10)}} \cdot \sum_{n=0}^{9} \mathrm{e}^{-j\theta\cdot n} = \frac{1}{\sqrt{(10)}} \cdot \frac{1-\mathrm{e}^{-j\theta\cdot 10}}{1-\mathrm{e}^{-j\theta}} \tag{3}$$

$$= \frac{1}{\sqrt{(10)}} \cdot \frac{\mathrm{e}^{-j\theta\cdot 5}}{\mathrm{e}^{-j\frac{\theta}{2}}} \cdot \frac{\mathrm{e}^{j\theta\cdot 5}-\mathrm{e}^{-j\theta\cdot 5}}{\mathrm{e}^{j\frac{\theta}{2}}-\mathrm{e}^{-j\theta/2}} \tag{4}$$

$$= \frac{1}{\sqrt{(10)}} \cdot \frac{\mathrm{e}^{-j\theta\cdot 5}}{\mathrm{e}^{-j\frac{\theta}{2}}} \cdot \frac{\sin(5\theta)}{\sin(\frac{\theta}{2})} \tag{5}$$




$$x[n-9] = x_r[n] * x_r[n] \circ\!\!-\!\!\bullet\; X_r\left(\mathrm{e}^{j\theta}\right) \cdot X_r\left(\mathrm{e}^{j\theta}\right) \tag{6}$$

$$= \frac{1}{10} \cdot \frac{\mathrm{e}^{-j\theta \cdot 10}}{\mathrm{e}^{-j\theta}} \cdot \left[\frac{\sin(5\theta)}{\sin(\frac{\theta}{2})}\right]^2 \tag{7}$$

$$= \frac{1}{10} \cdot \mathrm{e}^{-j\theta \cdot 9} \cdot \left[\frac{\sin(5\theta)}{\sin(\frac{\theta}{2})}\right]^2 \tag{8}$$

Shifting this to the left by 9 samples so that the triangle is centered again directly leads to the result:

$$X\left(\mathrm{e}^{j\theta}\right) = \mathrm{e}^{j\theta \cdot 9} \cdot X_r\left(\mathrm{e}^{j\theta}\right) \cdot X_r\left(\mathrm{e}^{j\theta}\right) \tag{9}$$

$$= \frac{1}{10} \cdot \left[\frac{\sin(5\theta)}{\sin(\frac{\theta}{2})}\right]^2 \tag{10}$$

## II. Plain Math Solution

$$X_r\left(\mathrm{e}^{j\theta}\right)\cdot X_r\left(\mathrm{e}^{j\theta}\right) = \frac{1}{10}\left[\frac{1}{1-\mathrm{e}^{-j\theta}}+\pi\delta_{2\pi}(\theta)-\frac{\mathrm{e}^{-j\theta\cdot 10}}{1-\mathrm{e}^{-j\theta}}-\pi\delta_{2\pi}(\theta)\cdot\mathrm{e}^{-j\theta\cdot 10}\right]^2 \quad (11)$$

$$= \frac{1}{10}\left[\frac{1}{1-\mathrm{e}^{-j\theta}}-\frac{\mathrm{e}^{-j\theta\cdot 10}}{1-\mathrm{e}^{-j\theta}}+\pi\underbrace{\left(\delta_{2\pi}(\theta)-\delta_{2\pi}(\theta)\cdot\mathrm{e}^{-j\theta\cdot 10}\right)}_{\delta\neq 0 \text{ iff } \theta=0}\right]^2 \quad (12)$$

$$= \frac{1}{10}\left[\frac{1}{1-\mathrm{e}^{-j\theta}}-\frac{\mathrm{e}^{-j\theta\cdot 10}}{1-\mathrm{e}^{-j\theta}}\right]^2 \quad (13)$$

$$= \frac{1}{10}\left[\frac{1-\mathrm{e}^{-j\theta\cdot 10}}{1-\mathrm{e}^{-j\theta}}\right]^2 \quad (14)$$

$$= \frac{1}{10}\left[\frac{1-2\mathrm{e}^{-j\theta\cdot 10}+\mathrm{e}^{-j\theta\cdot 20}}{1-2\mathrm{e}^{-j\theta\cdot 1}+\mathrm{e}^{-j\theta\cdot 2}}\right] \quad (15)$$

Shifting the last expression to the left by $9$ samples in the *frequency domain* (so that the triangle does not start at $0$ but at $-9$ leads to:

$$X_r\left(\mathrm{e}^{j\theta}\right)\cdot X_r\left(\mathrm{e}^{j\theta}\right)\cdot\mathrm{e}^{j\theta\cdot 9} = \frac{1}{10}\left[\frac{1-2\mathrm{e}^{-j\theta\cdot 10}+\mathrm{e}^{-j\theta\cdot 20}}{1-2\mathrm{e}^{-j\theta\cdot 1}+\mathrm{e}^{-j\theta\cdot 2}}\right]\cdot\mathrm{e}^{j\theta\cdot 9} \quad (16)$$

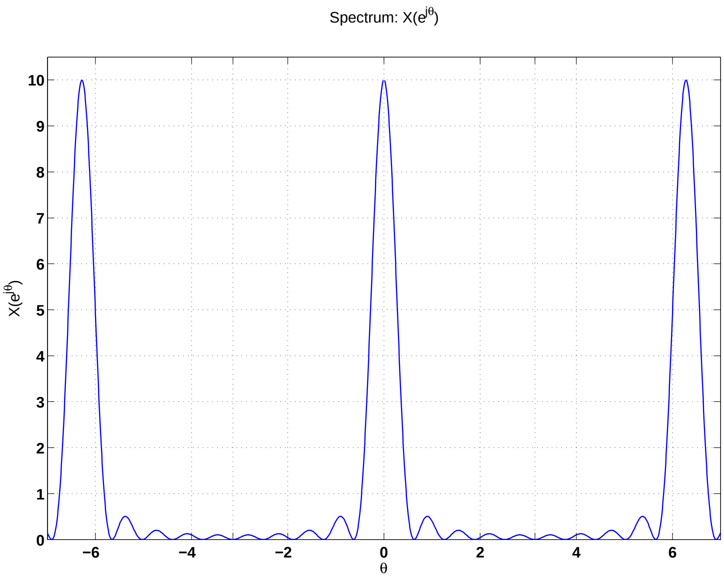
$$= \frac{1}{10}\left[\frac{\left(\mathrm{e}^{j\cdot 0}-2\mathrm{e}^{-j\theta\cdot 10}+\mathrm{e}^{-j\theta\cdot 20}\right)\cdot\mathrm{e}^{j\theta\cdot 10}}{(\mathrm{e}^{j\cdot 0}-2\mathrm{e}^{-j\theta\cdot 1}+\mathrm{e}^{-j\theta\cdot 2})\cdot\mathrm{e}^{j\theta\cdot 1}}\right] \quad (17)$$

$$= \frac{1}{10}\left[\frac{\mathrm{e}^{j\theta\cdot 10}-2\mathrm{e}^{-j\theta\cdot 0}+\mathrm{e}^{-j\theta\cdot 10}}{\mathrm{e}^{j\theta\cdot 1}-2\mathrm{e}^{-j\theta\cdot 0}+\mathrm{e}^{-j\theta\cdot 1}}\right] \quad (18)$$

$$= \frac{1}{10}\left[\frac{\mathrm{e}^{j\theta\cdot 5}-\mathrm{e}^{-j\theta\cdot 5}}{\mathrm{e}^{j\frac{\theta}{2}}-\mathrm{e}^{-j\frac{\theta}{2}}}\right]^2 \quad (19)$$

$$= \frac{1}{10}\left[\frac{\frac{1}{2j}\left(\mathrm{e}^{j\theta\cdot 5}-\mathrm{e}^{-j\theta\cdot 5}\right)}{\frac{1}{2j}\left(\mathrm{e}^{j\frac{\theta}{2}}-\mathrm{e}^{-j\frac{\theta}{2}}\right)}\right]^2 \quad (20)$$

$$= \frac{1}{10}\left[\frac{\sin(5\theta)}{\sin(\frac{\theta}{2})}\right]^2 \quad (21)$$


Spectrum: X(e^{jθ})
X(e^{jθ})
θ
0
1
2
3
4
5
6
7
8
9
10
−6
−4
−2
0
2
4
6