Digital Communications 2
Written exam on October 24, 2013

Institute of Telecommunications
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Please note:

- You are allowed to use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- You are not allowed to use handwritten lecture notes, materials from exercise classes, and precalculated problems.
- Readable handwriting and a clear presentation are mandatory.
- Provide detailed derivations and calculations. If results from the lecture notes are used, they have to be referenced unambiguously.
- Due to changes of the modus operandi for the exercise class since the 08/09 winter semester, the maximum number of achievable credits will be scaled from 80 to 90 for the corresponding candidates.

Problem 1 (20 credits)

Consider a binary Hamming code of length $N = 15$.

a) The codewords are transmitted over a BSC with bit-inversion probability $p = 0.02$.

a1) How many errors can the blockwise ML decoder detect and correct?

a2) Calculate the block error probability of the ML decoder. Compare your result with the block error probability obtained for transmission without coding.

a3) How many erasures can the ML decoder correct (in case of transmission over an error-and-erasure channel)?

b) Find a systematic generator matrix and check matrix.

c) The Hamming code is a cyclic code; its generator polynomial is $g(x) = x^4 + x + 1$. Calculate the syndrome polynomial associated with the received polynomial $v(x) = x^{10} + x^4 + x + 1$.

Problem 2 (20 credits)

Consider the binary linear encoder

$$(u_0\ u_1)^T \longrightarrow (u_0\ u_1\ u_0\ x)^T.$$

a) Choose x as a linear function of u_0 and u_1 such that the encoding results in a cyclic code $\mathcal{C}'$.

b) Consider the extension of $\mathcal{C}'$ by a single parity bit. Does this code provide any gain in comparison to $\mathcal{C}'$ (in the sense of number of detectable/correctable errors)? Explain!

c) Choose x as a linear function of u_0 and u_1 such that the encoding results in a non-cyclic code $\mathcal{C}$. Determine the parameters N, K, and $d_{\min}$.

d) Find a generator matrix $\mathbf{G}$ and a check matrix $\mathbf{H}$ of $\mathcal{C}$.

e) Consider transmission of $\mathcal{C}$ over a binary symmetric error-and-erasure channel with bit-inversion probability $p = 0.1$ and erasure probability $p_e = 0.2$. The decoder consists of the following two stages: the first stage replaces each erasure by 0 or by 1 with equal probability, while the second stage is a bounded-minimum-distance decoder with maximum decoding radius. Calculate the resulting block error probability of the decoder. Compare your result with the block error probability of uncoded transmission followed by the erasure-filling stage.

Problem 3 (20 credits)

Consider the set of all cyclic codes over $\mathrm{GF}(3) = \{0, 1, 2\}$ with block length $N = 4$.

a) Provide the addition and multiplication tables of GF(3).

b) Specify all generator polynomials and the corresponding code rates (if necessary, express higher-order polynomials with the help of other generator polynomials).

c) Choose a generator polynomial with the highest possible rate and check whether $v(x) = 2x^2 + 1$ is a valid codeword.

d) For the code used in c), find the generator polynomial of the dual code.

e) Draw the shift-register circuit of the nonsystematic encoder for the code in c).

Problem 4 (20 credits)

This problem investigates the Reed-Solomon code $\mathcal{C}$ with block length $N = 15$, minimum distance $d_{\min} = 13$, and frequency parameter $k_1 = 3$.

a) Calculate the rate R.

b) Which Galois field GF(q) is the basis of this code?

c) Give an expression for the generator polynomial $g(x)$ and for the check polynomial $h(x)$ in terms of a primitive element $z \in \mathrm{GF}(q)$.

d) Perform systematic encoding of the dataword $\mathbf{u} = (z^0\, z^5\, z^1)^T$ in the frequency domain. Specify the resulting codeword in the frequency domain and in the time domain.

e) Check whether

$$\mathbf{v} = (z^1\, z^{12}\, z^5\, z^{13}\, z^1\, z^2\, z^{13}\, z^7\, z^{11}\, z^0\, z^1\, z^{10}\, z^2\, z^8\, z^{13})^T$$

is a codeword of $\mathcal{C}$. Hint:

$$\mathcal{F}^{-1}\left\{(0\;0\;0\;0\;0\;z^0\;z^5\;z^1\;0\;0\;0\;0\;0\;0\;0)^T\right\} = (z^8\, z^{13}\, z^1\, z^{12}\, z^5\, z^{13}\, z^1\, z^2\, z^{13}\, z^7\, z^{11}\, z^0\, z^1\, z^{10}\, z^2)^T.$$