Digital Communications 2
Written exam on April 30, 2014

Institute of Telecommunications
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Please note:

- You are allowed to use the official lecture notes, a pocket calculator, and a collection of mathematical formulas.
- You are not allowed to use handwritten lecture notes, materials from exercise classes, and precalculated problems.
- Readable handwriting and a clear presentation are mandatory.
- Provide detailed derivations and calculations. If results from the lecture notes are used, they have to be referenced unambiguously.
- Due to changes of the modus operandi for the exercise class since the 08/09 winter semester, the maximum number of achievable credits will be scaled from 80 to 90 for the corresponding candidates.

Problem 1 (20 credits)

Consider a Hamming code $\mathcal{C}$ of length $N = 15$.

a) The codewords are transmitted over a BSC with bit-inversion probability $p = 0.01$.

a1) How many errors are allowed to occur such that the blockwise ML decoder decodes correctly?

a2) Calculate the block error probability of the ML decoder. Compare the result to the block error probability of uncoded transmission.

b) Determine a systematic generator matrix and a systematic check matrix.

c) Determine the check matrix $\mathbf{H}^{\perp}$ of the dual code $\mathcal{C}^{\perp}$. What is the minimum Hamming distance of $\mathcal{C}^{\perp}$?

Problem 2 (20 credits)

The Galois field $\mathrm{GF}(8) = \{0, z^0, \ldots, z^6\}$ is generated by the primitive polynomial $f(x) = x^3 + x + 1$ over GF(2). The element $z \in \mathrm{GF}(8)$ is a primitive element, i.e., $f(z) = 0$.

a) Derive a polynomial representation over GF(2) of the elements of GF(8).

b) Decompose $f(x)$ into linear factors over GF(8).

c) Show that $f(x)$ is the generator polynomial of a primitive cyclic code over GF(2) with block length $N = 7$. What is the rate of this code? Using the result of b), check whether $v(x) = x^6 + x^5 + x^2 + 1$ is a valid code polynomial.

Problem 3 (20 credits)

Consider the binary linear block code $\mathcal{C}$ with check matrix

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

a) Determine N, K, and $d_{\min}$. Construct a systematic generator matrix $\mathbf{G}$ for $\mathcal{C}$ (not for an equivalent code). Is $\mathcal{C}$ a cyclic code?

b) Calculate the parameters N', K', and $d'_{\min}$, and construct a generator matrix $\mathbf{G}'$ and a check matrix $\mathbf{H}'$ for the codes $\mathcal{C}'$ obtained by the following modifications:

b1) Extending:

$$(u_0\ u_1\ c_2\ c_3\ c_4)^T \in \mathcal{C} \quad \rightarrow \quad (u_0\ u_1\ c_2\ c_3\ c_4\ u_0{+}u_1{+}c_2{+}c_4)^T \in \mathcal{C}'\,.$$

b2) Puncturing:

$$(u_0\ u_1\ c_2\ c_3\ c_4)^T \in \mathcal{C} \quad \rightarrow \quad (u_0\ u_1\ c_2\ c_4)^T \in \mathcal{C}'\,.$$

b3) Lengthening:

$$(u_0\ u_1\ c_2\ c_3\ c_4)^T \in \mathcal{C} \quad \rightarrow \quad (u_0\ u_1\ u_2\ c_2{+}u_2\ c_3{+}u_2\ c_4)^T \in \mathcal{C}'\,.$$

b4) Shortening:

$$(u_0\ u_1\ c_2\ c_3\ c_4)^T \in \mathcal{C} \quad \rightarrow \quad (u_0\ c'_1\ c'_2\ c'_3)^T \in \mathcal{C}'\,,$$

with

$$\begin{pmatrix} c'_1 \\ c'_2 \\ c'_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{G} \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$$

Problem 4 (20 credits)

a) Show that for a $(N, K)_q$ linear block code with minimum Hamming distance at least $2t+1$

$$q^{N-K} \geq \sum_{i=0}^{t} \binom{N}{i} q^i \,.$$

b) Show that equality holds for perfect $(N, K)_q$ linear block codes.