Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 3.1 (0.5 points)
Let X be the set of all sequences with l_2-norm equal to $\sqrt{2}$, i.e.,

$$\left\{ x \in X : \left(\sum_{n=1}^{\infty} x_n^2 \right)^{\frac{1}{2}} = \sqrt{2} \right\}, \tag{1}$$

where x_n denotes the n-th element of sequence x. Consider

$$d = \sum_{n=1}^{\infty} x_n y_n \tag{2}$$

with $\{x, y\} \in X$.

1. *Do X and d constitute a metric space (X,d)?*
2. *Calculate tight upper and lower bounds of d.*
3. *Consider $\{x, y\} \in X$ and the sequence y is given. Find the sequences x, for which the lower and upper bounds of the previous point are achieved.*
4. *Calculate the bounds for*

$$\sum_{n=1}^{\infty} (x_n - y_n)^2 \pm \sum_{n=1}^{\infty} (x_n + y_n)^2, \quad \{x, y\} \in X \tag{3}$$

5. *Assume the sequences $\{x, y\} \in X$ to be given as*

$$x_n = \sin\left(\frac{\pi}{2} n\right)(u_n - u_{n-4}) \qquad n = 1,2,\ldots,\infty \tag{4}$$

$$y_n = a_1 \delta_{n-1} + a_2 \delta_{n-2} + a_3 \delta_{n-3} \qquad n = 1,2,\ldots,\infty \tag{5}$$

with $\{a_1, a_2, a_3\} \in \mathbb{R}$. *The function* u_n *denotes the unit step function, i.e.,*

$$u_n = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}, \tag{6}$$

and δ_n *is the delta function.*
Calculate a_1, a_2 *and* a_3 *such that* d *(as given in (2)) becomes zero. Determine the number of possible solutions for* $\{a_1, a_2, a_3\}$.

Exercise 3.2 (0.5 points)
The convergence / divergence of a series can be checked by one of the following theorems (without claiming to be exhaustive):

(I) *Let* $\sum a_k$ *and* $\sum b_k$ *be series with* $a_k, b_k \in \mathbb{R}^+$. *Given that* $a_k \leq b_k \forall k$, *then*

- *If* $\sum b_k$ *converges* $\Rightarrow \sum a_k$ *converges*
- *If* $\sum a_k$ *diverges* $\Rightarrow \sum b_k$ *diverges*

(II) *Let* $\sum a_k$ *and* $\sum b_k$ *be a series with* $a_k, b_k \in \mathbb{R}^+$ *and* $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$. *If* ρ *is finite and* $\rho \in \mathbb{R}^+ \backslash 0$, *then the series either both converge or both diverge.*

(III) *Let* $\sum a_k$ *be a series with* $a_k \in \mathbb{R}^+$. *Then for either* $\rho = \lim_{k\to\infty} (a_k)^{\frac{1}{k}}$, *or* $\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ *it holds*

- *if* $\rho < 1 \Rightarrow$ *series converges*
- *if* $\rho = 1 \Rightarrow$ *no statement about convergence / divergence can be made*
- *if* $\rho > 1$ *or* $q = +\infty \Rightarrow$ *series diverges*

(IV) *Let f(x) be a monotonically decreasing, non-negative function with support* $[N,\infty)$. *The series* $\sum_{k=N}^{\infty} f(k)$ *converges to a real number iff* $\int_N^{\infty} f(x)dx$ *is finite.*

(V) *Let a series be given by* $\sum \frac{1}{k^p}$. *Then,*

- *if* $0 < p \leq 1 \Rightarrow$ *series diverges*
- *if* $p > 1 \Rightarrow$ *series converges*

Apply the theorems given above to check the convergence / divergence of the following series:

1. $\sum_{k=1}^{\infty} \frac{(2k)^{k+2}}{(k+1)!}$
2. $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}-1}$
3. $\sum_{k=1}^{\infty} \left[k^4 \sin^2\left(\frac{3k}{2k^3-2k^2+5}\right)\right]^k$
4. $\sum_{k=1}^{\infty} \frac{(3k)!+4^{k+1}}{(3k+1)!}$

5. ***Extra 0.2 points***: $\sum_{k=2}^{\infty} \frac{1}{k \log k}$
   Hint: Use the fact that $f(k+1) \leq f(x) \leq f(k)$ *implies* $f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k)$.

Exercise 3.3 (0.5 points)

A common method for optimal resource allocation in mobile cellular networks is convex optimization. A convex function satisfies the following property:

$$f(\theta s + (1-\theta)t) \leq \theta f(s) + (1-\theta) f(t) \tag{7}$$

where $0 \leq \theta \leq 1$.

1. *Assume* $f : \mathbb{R}^n \to \mathbb{R}$ *is an arbitrary norm. Show that every norm on* $\mathbb{R}^n$ *is convex.*

2. *Show that the following expressions define norms on* $\mathbb{R}^n$*, with* $\underline{x} = (x_1, x_2, \ldots, x_n)^T$*:*

- $\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$,
- $\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$,
- $\|\underline{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

MATLAB-Exercise 3.1 (1.5 points)
Consider a linear time-variant filter with linear phase

$$H(e^{j\Omega}) = a_0 + \sum_{k=1}^{N} 2a_k \cos(k\Omega) \,. \tag{8}$$

1. *Analytical part: Determine the coefficients of this filter such that it optimally follows*

$$H^{(d)}\left(e^{j\Omega}\right) = \begin{cases} 0.2 & ,|\Omega| < \Omega_1 \\ 1 & ,\Omega_1 \le |\Omega| \le \Omega_2 \\ 0.2 & ,|\Omega| < \pi \end{cases} \tag{9}$$

   in the Least Squares (LS) sense. This is equivalent to minimizing the metric

$$d_2(H^{(d)}(e^{j\Omega}), H(e^{j\Omega})) = \int_{-\pi}^{\pi} \left| H^{(d)}\left(e^{j\Omega}\right) - H(e^{j\Omega}) \right|^2 d\Omega. \tag{10}$$

   Hints:

   - *Apply the Parseval theorem to get an equivalent statement in the time domain.*
   - *Write the desired transfer function as a linear combination of ideal low-pass filters.*
   - *To minimize a convex function, you can differentiate it with respect to the variables and set the derivative equal to zero.*

2. *MATLAB part:*

   (a) *Implement a MATLAB code to plot the frequency response of the filter $H(e^{j\Omega})$, as obtained from your analytic calculations for arbitrary filter order. The limit frequencies are $\Omega_1 = \frac{\pi}{4}$ and $\Omega_2 = \frac{3\pi}{4}$.*

   (b) *Compare the filters for the orders $N = \{4, 10, 100\}$. What do you observe?*