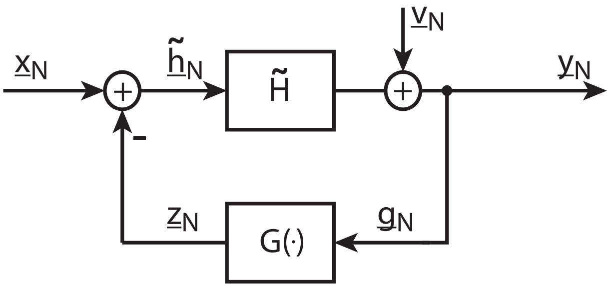
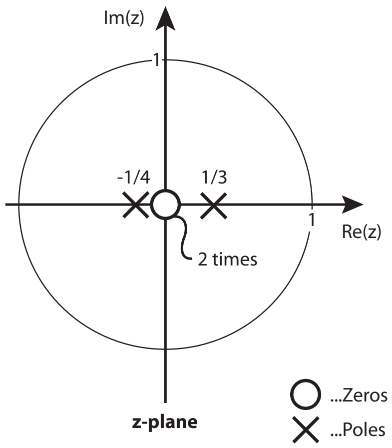
Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.



(a) Signal-flow diagram.



(b) Pole-Zero plot of the system $\tilde{H}(z)$.

Figure 4.1: Feedback system

Exercise 4.1 (0.5 points)
Consider the feedback system shown in Figure 4.1a. The causal system $\tilde{H}(z)$ is given by its pole-zero diagram shown in Figure 4.1b. It has two poles and two zeros. The system G is given by the non-linear relationship

$$G(x) = \frac{x}{\sqrt{e^{-ax} + e^{bx}}},\ a,b \in \mathbb{R}. \quad (1)$$

1. *(0.2 points) Calculate the gain of the system $\tilde{H}(z)$.*
2. *(0.2 points) Calculate the gain of the system $G(\cdot)$.*

(Hint: Consider the l_2 norm.)
From now on we set $a = 1$ and $b = 1$.

3. *(0.1 points) Assuming the additive disturbance* $v_n = 0\ \forall n$*, is the feedback system stable? Justify your answer.*

4. *(0.25 points) Assuming zero initial conditions for the systems* $\tilde{H}$ *and* G*, derive an upper bound on* $||\underline{y}_N||_2$ *in terms of* $\underline{x}_N$*,* $\underline{v}_N$ *and the gains of the systems (*$\underline{y}_N$ *is the length* N *vector of output samples).*

Exercise 4.2 (0.5 points)
In this exercise we investigate the transmission of a signal with frequency offset ϵ. The received signal is expressed as

$$r_k = s_k\, h\, e^{j\epsilon k} + v_k, \tag{2}$$

where s_k denotes the pilot symbol, which is known at the receiver. It stays constant over time, i.e., $s_k = s, \forall k$, and $|s|^2 = \sigma_s^2$. The complex-valued channel h is also assumed to be static over time. Frequency offset ϵ introduces a periodic change in phase. The additive noise terms v_k are assumed to be independent and identically distributed (i.i.d.) complex Gaussian with variance σ_v^2, i.e., $v_k \sim \mathcal{CN}(0,\sigma_v^2)$.
Your task is to estimate the frequency offset at the receiver:

1. *Calculate $y_k = r_k r_{k+1}^*$ as an observation for determining the frequency offset. Identify the signal- and noise terms, respectively.*
2. *Let $y_k = ce^{-j\epsilon} + w_k$, where w_k denotes the noise terms. Based on the previous point, calculate the Signal to Noise Ratio (SNR) $\gamma = \mathbb{E}[|c|^2]/\mathbb{E}[|w_k|^2]$. (Hint: w_k contains all terms with exponentials different from $e^{-j\epsilon}$)*
3. *Considering only the signal term, determine the frequency offset ϵ.*
4. *Next, we assume to receive M signals*

$$r_k^{(m)} = s_m h_m e^{j\epsilon k} + v_k^{(m)}, \quad m = 1,2,...M, \tag{3}$$

at the same time, where the symbols s_m are statistically independent. The complex-valued channel attenuations h_m are all different, $\mathbb{E}\{|v_k^{(m)}|^2\} = \sigma_v^2$ and $|s_m|^2 = \sigma_s^2$, $\forall m$. We compute the M observations $y_k^{(m)}$ and equally combine them, which is written as $\sum_{m=1}^{M} y_k^{(m)}$. Calculate the SNR after combining.

5. *Now we combine the M observations $y_k^{(m)}$ using weighting factors g_m, i.e., $\sum_{m=1}^{M} g_m y_k^{(m)}$. Determine the g_m such that the SNR after the combiner is maximized. What do you obtain for the SNR?*
Hint: Try to rewrite the expression by considering the term
$p_m = g_m\sqrt{2\sigma_s^2|h_m|^2 + \sigma_v^2}$.

Exercise 4.3 (0.5 points)

- *Proof that*

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \tag{4}$$

is a metric, where x_n *and* y_n *are from the vector space of real-valued sequences*

$$\mathbb{R}^{\mathbb{N}} = \{x = (x_n)_{n\in\mathbb{N}} | x_n \in \mathbb{R}\}. \tag{5}$$

- *A norm is induced by an inner product iff it satisfies*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{6}$$

Prove this identity and determine, if L_2 *is a Hilbert space with inner product* $\langle f,g\rangle = \int \bar{f}(x)g(x)dx$.

- *Show that* $d(x,y) = |\arctan(x) - \arctan(y)|$ *is a metric on* $\mathbb{R}$ *and determine if* $(\mathbb{R},d)$ *constitutes a complete metric space.*

MATLAB-Exercise 4.1 (1.5 points)
In this MATLAB exercise, we evaluate the frequency-offset problem from Exercise 4.2 by simulations. The received signal is expressed as

$$r_k = h\, e^{j\epsilon k} + v_k, \tag{7}$$

i.e., the all-one sequence is applied for the training symbols.
Consider a frequency offset $\epsilon = \pi/4$. *Your task is to determine and plot the average estimation error* $|\hat{\epsilon} - \epsilon|^2$ *over* 2000 *simulation runs for an SNR range* $-20\,\mathrm{dB} \ldots 20\,\mathrm{dB}$. *For each simulation run, use* `randn()` *to generate the complex valued channel(s) and the noise, and norm them properly.*
The following cases should be considered:

1. *Single channel*

   Compute the average frequency offset ϵ *based on one observation* y_k *by* `-angle`(y_k).

2. *Single channel, multiple observations*

   Consider that we receive 5 *symbols* r_k, $k = 1, \ldots, 5$, *from a single channel. This results in* 4 *instantaneous observations* y_k, $k = 1, \ldots, 4$. *Compute the frequency offset* ϵ *by* `-angle`$(\sum_{k=1}^{4} y_k)$.

3. *Multiple channels, equal weight combining*

   Repeat the experiment by using four independent channels simultaneously $\{h_1, h_2, h_3, h_4\}$. *Still, all-one sequences are transmitted on all the channels. At each time instant* k, *four observations* ($y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, y_k^{(4)}$) *are available. Compute the frequency offset* ϵ *using the equal weight combining by* `-angle`$(\sum_{m=1}^{4} y_k^{(m)})$.

4. *Multiple channels, SNR-optimal combining*

   Repeat the four-channel experiment. Employ the weighting factors, as derived in Exercise 4.2/5 and compute the frequency offset by `-angle`$(\sum_{m=1}^{4} g_m y_k^{(m)})$. *Compare the result with the equal weight combining.*