Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 5.1 (0.75 points)

Consider the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_4(t)$ of Figure 5.1.

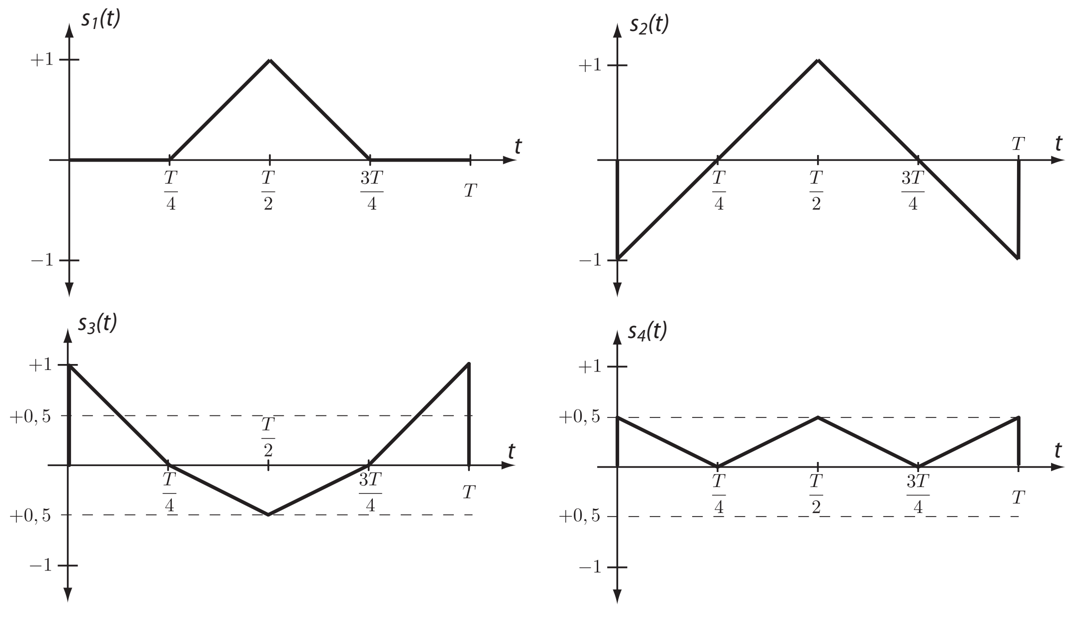

Figure 5.1: Signals $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_4(t)$.

1. *(0.15 points) Evaluate $\|s_1\|_2^2$, $\|s_2\|_2^2$, $\|s_3\|_2^2$ and $\|s_4\|_2^2$, with*

$$\|x\|_2 = \sqrt{\int_0^T x(t)^2 dt}. \tag{1}$$

2. *(0.15 points) Derive an orthonormal basis $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, $\phi_4(t)$ for the space spanned by $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_4(t)$. Use the Gram-Schmidt orthogonalization method and start with $s_1(t)$. Sketch the evaluated basis functions. Which dimension has the signal space?*

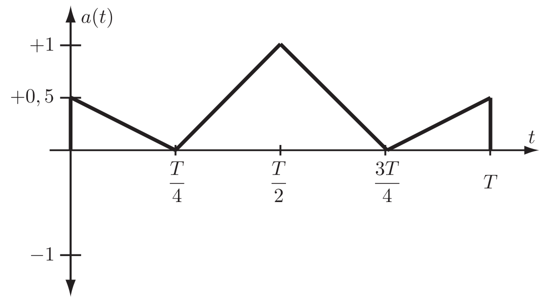

Figure 5.2: Signal $a(t)$.

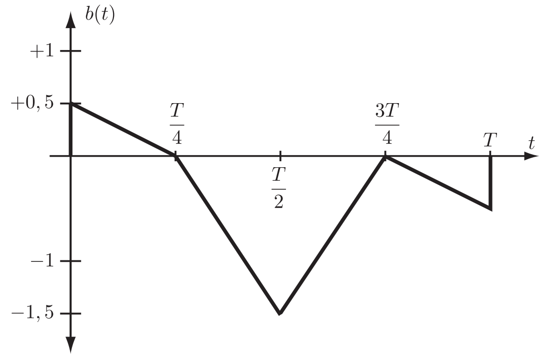

Figure 5.3: Signal $b(t)$.

3. *(0.15 points) Now consider the signal $a(t)$ in Figure 5.2. Express the signal in terms of the derived basis and calculate $\|a - s_3\|_2^2$.*

4. *(0.15 points) Finally consider the signal $b(t)$ in Figure 5.3. Find the best approximation $\hat{b}(t)$ of this signal in the space spanned by the basis $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, $\phi_4(t)$ in terms of the L_2 norm.*

5. *(0.15 points) Sketch $\hat{b}(t)$.*

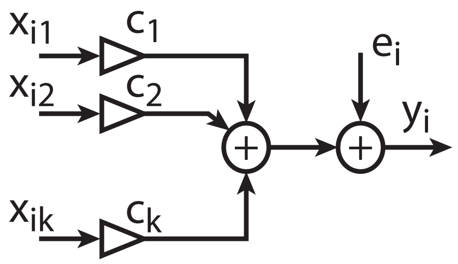

Figure 5.4: Linear masurement model

Exercise 5.2 (0.75 points)

Least squares (LS) regression and best linear unbiased estimation

In this exercise we consider linear least squares regression and derive some important statistical properties of the least squares estimator.

Consider the general linear measurement model visualized in 5.4 of the form

$$\underline{y} = X\underline{c} + \underline{e}, \quad X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & & \vdots \\ x_{p1} & x_{12} & \dots & x_{pk} \end{bmatrix} \tag{2}$$

where $\underline{y} \in \mathbb{C}^{p\times 1}$ *is the measured data,* $\underline{c} \in \mathbb{C}^{k\times 1}$ *are unobservable parameters* $(k \leq p)$*,* $X \in \mathbb{C}^{p\times k}$ *is the input data of the measurement system and* $\underline{e}$ *is the measurement error (the input data is selected such that* $\operatorname{rank}(X) = k$*). We want to apply the linear regression model*

$$\hat{\underline{y}} = X\hat{\underline{c}} \tag{3}$$

to this measurement model.

1. *(0.05 points) Apply an ordinary least squares estimation to determine the parameters* $\hat{\underline{c}}$.
2. *(0.1 points) Assuming a zero-mean error* $\mathbb{E}(\underline{e}) = \underline{0}$*, show that the obtained estimator is unbiased, i.e.,* $\mathbb{E}(\hat{\underline{c}}) = \underline{c}$.
3. *(0.1 points) Calculate the covariance of the estimator, i.e.,*

$$\operatorname{Cov}(\hat{\underline{c}}) = \mathbb{E}\left((\hat{\underline{c}} - \mathbb{E}(\hat{\underline{c}}))(\hat{\underline{c}} - \mathbb{E}(\hat{\underline{c}}))^{\mathrm{H}}\right) \tag{4}$$

using the results above.

Next, we want to show that our linear least squares estimator is the best linear unbiased estimator (BLUE) for the parameters of the model in Equation (2), in the sense that it minimizes the mean squared error $\mathbb{E}(\|\hat{\underline{c}} - \underline{c}\|^2)$*, for the case that* $\mathbb{E}(\underline{e}) = \underline{0}$, $\mathbb{E}\left(\underline{e}\,\underline{e}^{\mathrm{H}}\right) = \sigma_e^2 I$.

4. *(0.2 points) Show that the mean squared error can be written as*

$$\mathbb{E}\left(\|\hat{\underline{c}} - \underline{c}\|^2\right) = \mathrm{trace}\left(\mathrm{Cov}(\hat{\underline{c}})\right) + \|\mathrm{Bias}(\hat{\underline{c}},\underline{c})\|^2, \tag{5}$$
$$\mathrm{Bias}(\hat{\underline{c}},\underline{c}) = \mathbb{E}\left(\hat{\underline{c}}\right) - \underline{c}. \tag{6}$$

As we consider unbiased estimators, the bias-term above is zero. Hence, we can focus on the covariance matrix of the estimator. Suppose we have another linear unbiased estimator of the parameter vector $\underline{c}$ which we write as

$$\tilde{\underline{c}} = \hat{\underline{c}} + A\underline{y} = B\underline{y}. \tag{7}$$

5. *(0.1 points) Which condition does A have to satisfy for the estimator to be unbiased?*

6. *(0.1 points) Calculate the covariance matrix of $\tilde{\underline{c}}$.*

7. *(0.1 points) Show that* $\mathrm{Cov}(\tilde{\underline{c}})$ *exceeds* $\mathrm{Cov}(\hat{\underline{c}})$ *by a positive semidefinite matrix, and hence its trace is larger (implying a larger MSE).*

Exercise 5.3 (0.5 points)
Consider the linear differential equation

$$\frac{d}{dx}y(x) - y(x) = q(x) \tag{8}$$

We try to approximate the solution $y(x)$ by using orthogonal polynomials with highest order 3 as our basis functions. The polynomials, also known as Tschebyscheff polynomials, are given as:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

and $y(x)$ is approximated as $\hat{y}(x) = a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + a_3T_3(x)$.

1. *(0.1 point) Show that T_0 and T_1 are orthogonal on the interval $(-1,1)$ with respect to the weight $w(x) = \frac{1}{\sqrt{(1-x^2)}}$.*
2. *(0.2 points) Calculate the coefficients $a_0,\ldots,a_3$ such that $\hat{y}(x)$ is a solution of the* homogeneous *differential equation (i.e. $q(x) = 0$) in the space* $\mathrm{span}\{T_0(x),\ldots,T_3(x)\}$. *For this task, consider the initial condition $\hat{y}(0) = \frac{7}{9}$.*
3. *(0.2 points) Now consider $q(x) = x(2x+1)$. Determine the coefficients $a_0,\ldots,a_3$ such that $\underline{a} = [a_0\ a_1\ a_2\ a_3]^T$ is the minimum norm solution (with respect to $\|\underline{a}\|_2$) of the* inhomogeneous *differential equation as given in Equation (8). Note: In this task, you don't have to evaluate the coefficients numerically.*
4. ***(0.5 extra points)**: Numerically evaluate the coefficients $a_0,\ldots,a_3$ of the previous point.*

MATLAB-Exercise 5.1 (1.5 points)
Given the set of data points

$$x = \{2,3,4,5,6,8,9\}; \quad y = \{25,13,4,4,3,7,4\}. \tag{9}$$

1. *Depict the data in a 2-dimensional plot.*
2. *Determine the quadratic approximation $y = ax^2 + bx + c$, which best fits the data in the Least Squares (LS) sense.*
3. *Assume that the first- and the last data points are the most accurate ones. Formulate the corresponding weighting matrix and determine the quadratic approximation based on weighted LS.*