

Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 6.1 (0.5 points)

In this exercise we consider the isomorphism $(\cdot)_{\mathbb{R}} : \mathbb{C}^m \rightarrow \mathbb{R}^{2m}$ between the complex vector space \mathbb{C}^m and the real vector space \mathbb{R}^{2m}

$$(\underline{z})_{\mathbb{R}} := \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix}, \quad (1)$$

$$\underline{z} = \underline{x} + j\underline{y} \in \mathbb{C}^m, \quad (2)$$

$$\underline{x}, \underline{y} \in \mathbb{R}^m. \quad (3)$$

Furthermore, we define the mapping $\{\cdot\}_{\mathbb{R}} : \mathbb{C}^{n \times m} \rightarrow \mathbb{R}^{2n \times 2m}$

$$\{C\}_{\mathbb{R}} := \begin{bmatrix} A & -B \\ B & A \end{bmatrix}, \quad (4)$$

$$C = A + jB \in \mathbb{C}^{n \times m}, \quad (5)$$

$$A, B \in \mathbb{R}^{n \times m}. \quad (6)$$

The real matrix on the right hand side of (4) is said to be of **complex form**.

1. Express the vector $\underline{u} = (C\underline{z})_{\mathbb{R}}$ in terms of $\{C\}_{\mathbb{R}}$ and $(\underline{z})_{\mathbb{R}}$.
2. Show that the mapping $\{\cdot\}_{\mathbb{R}}$ is a linear operator. (Note: for us (engineers) the homogeneity property is restricted to the field of real valued numbers)
3. Consider the matrices $X \in \mathbb{C}^{n \times m}$ and $Y \in \mathbb{C}^{m \times l}$. Show that

$$\{XY\}_{\mathbb{R}} = \{X\}_{\mathbb{R}} \{Y\}_{\mathbb{R}}. \quad (7)$$

4. If $U \in \mathbb{C}^{m \times m}$ is (a) hermitian, (b) unitary. What does that mean for $\{U\}_{\mathbb{R}}$?
5. Prove that $U \in \mathbb{C}^{m \times m}$ is non-singular if and only if $\{U\}_{\mathbb{R}}$ is non-singular.
6. Show that $U \in \mathbb{C}^{m \times m}$ is a positive definite hermitian matrix if and only if $\{U\}_{\mathbb{R}}$ is positive definite symmetric.

Exercise 6.2 (0.5 points)

1. Let be $A : X \rightarrow Y$ a bounded linear operator between two Hilbert spaces X and Y and $R(A)$ as well as $R(A^H)$ complete subspaces. Then we have:

(a) $\{R(A)\}^\perp = N(A^H)$

(b) $R(A) = \{N(A^H)\}^\perp$

(c) $\{R(A^H)\}^\perp = N(A)$

(d) $R(A^H) = \{N(A)\}^\perp$

Proof the four statements.

2. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Consider the mapping $A : \mathbb{R}^2 \rightarrow \mathbb{R}^4$:

$$\underline{x} \in \mathbb{R}^2 \rightarrow \underline{y} \in \mathbb{R}^4 : \quad \underline{x} = \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \underline{y} = A(\underline{x}) = \begin{bmatrix} a \\ a - b \\ b \\ a + b \end{bmatrix}. \quad (8)$$

(a) Specify the subspace V spanned by the mapping $A(\underline{x})$.

(Hint: employ vector/matrix notation)

(b) Specify the orthogonal complement of V .

Exercise 6.3 (0.5 points)

The Frobenius norm of a matrix $A \in \mathbb{R}^{n \times p}$ is defined as

$$\|A\|_F = \sqrt{\text{tr}(A^H A)} = \sqrt{\sum_{i=1}^n \sum_{j=1}^p a_{ij}^2}. \quad (9)$$

1. Show (proof) that the following statements hold for $\|\cdot\|_F$

(a) $\|A\| \geq 0$.

(b) Let $B \in \mathbb{R}^{m \times n}$. Then, $\|BA\| \leq \|B\| \|A\|$.

(c) Let $C \in \mathbb{R}^{n \times p}$. Then, $\|A + C\| \leq \|A\| + \|C\|$.

(d) Let $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{p \times p}$ denote unitary matrices. Then, $\|UAV\| = \|A\|$.

2. Given three possible alternative definitions of the norm of A ,

(a) $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$,

(b) $\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}(A)$ denotes the largest singular value of A , and

(c) $\|A\|_3 = \max_i \sum_{j=1}^p |a_{ij}|$.

Do these norms satisfy the statements 1.(a) – 1.(d)? Justify your answer.

MATLAB-Exercise 6.1 (1.5 points)

In this example we consider multi-user MIMO transmission in a MIMO broadcast system as shown in Fig. (6.1). In this broadcast system, there is one base station

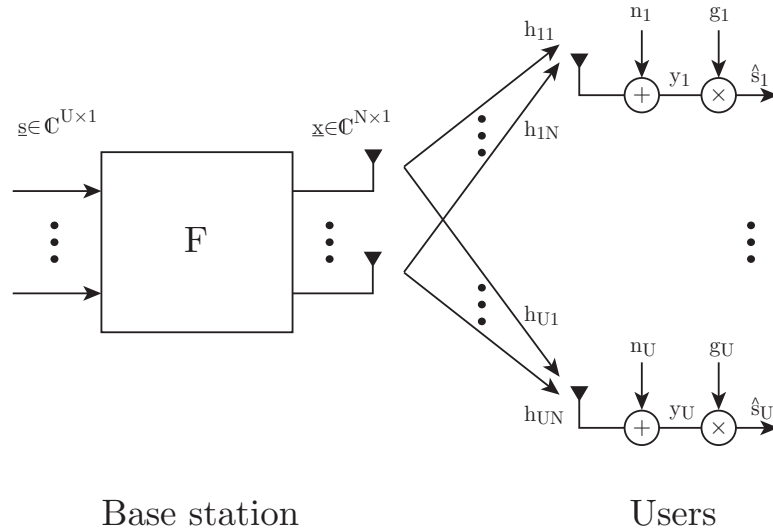


Figure 6.1: MU-MIMO transmission in a broadcast system

with N transmit antennas which serves U users in parallel, each with one receive antenna. The base station transmits a symbol s_u to each user $u \in \{1, \dots, U\}$. All transmit symbols are combined in the vector $\underline{s} = [s_1, \dots, s_U]^T$. Before transmission, the symbols are multiplied with the matrix $F = [\underline{f}_1, \dots, \underline{f}_U] \in \mathbb{C}^{N \times U}$ producing the vector \underline{x} . This operation is called precoding.

User u receives the transmit symbols over the vector channel $\underline{h}_u = [h_{u1}, \dots, h_{uN}]^T$. (Notice: for convenience we write the channel as a column vector). The observed channel output y_u is disturbed by zero mean additive white complex Gaussian noise of variance σ_u^2 , $n_u \sim \mathcal{N}(0, \sigma_u^2)$. The user multiplies the channel output y_u with the scalar g_u to obtain an estimate \hat{s}_u of his transmit symbol s_u . Notice that the user u in general also receives the symbols of the other users $v \neq u$.

Analytical part:

- Formulate the input-output relationship of user u between the transmit symbols $s_i, i \in \{1, \dots, U\}$ and the scalar channel output y_u . Partition the equation into useful signal, multi-user interference (interference from other users' symbols s_v) and noise.
- Considering the input-output relationship derived in the previous point, how do you have to select the precoders \underline{f}_v of the interfering users $v \neq u$ such that the multi-user interference observed by user u vanishes, i.e., $\underline{h}_u^T \underline{f}_v =$

0, $\forall v \neq u$? Think about the fundamental spaces (range, kernel) associated with the channel vector \underline{h}_u . (The trivial solution $\underline{f}_v = \underline{0}$ does not count).

- c) With the construction from the previous point, user u does not perceive any interference. Now we want to achieve the same for all U users. For this purpose we consider user u again. Combine the channels of all other users $v \neq u$ into the matrix $\bar{H}_u = [\underline{h}_1, \dots, \underline{h}_{u-1}, \underline{h}_{u+1}, \dots, \underline{h}_U]$. Assuming \bar{H}_u to be full rank, how do you have to select the precoder \underline{f}_u of user u such that all other users $v \neq u$ do not perceive any interference from user u , i.e., $\bar{H}_u^T \underline{f}_u = \underline{0}$? (Again, we want the non-trivial solution). What is the maximum number of users U_{max} that the system can serve in parallel interference free? Formulate your solution for the precoder in terms of the fundamental spaces of \bar{H}_u^T .

Matlab part:

1. Implement the system shown in Fig. (6.1) in Matlab assuming $N = 4$. Select the symbols s_u from a 4 QAM modulation alphabet normalized to unit power. The channels h_{ij} are independently generated from a complex Gaussian distribution of variance 1, i.e., $h_{ij} \sim \mathcal{N}(0,1)$. Assume equal noise power $\sigma_u^2 = \sigma_n^2$ for all users. Calculate the equalizers as

$$g_u = \frac{1}{\underline{h}_u^T \underline{f}_u}. \quad (10)$$

Use a minimum distance quantizer to map the equalized symbols onto the nearest 4 QAM constellation point.

2. Select the maximum number of users as obtained from point c) and calculate the precoders according to your solution from point c). The precoding vectors should have unit norm. Employ the SVD to determine the fundamental spaces of the matrix \bar{H}_u^T , required for the calculation of \underline{f}_u . (If you were not able to solve the analytical part, you can alternatively set $U = 2$ and select the precoders randomly to get half of the points).
3. Plot the average symbol error ratio (SER) versus the SNR = $\frac{1}{\sigma_n^2}$ in the range $SNR \in [-10, 20]$ dB. Simulate 5000 channel and noise realizations per SNR value and average the SER over channel and noise realizations as well as over users.