Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 7.1 (0.5 points)
Consider a set of Hermitian matrices $A^{(i)}$, $i = 1, \dots, N$ of dimension $M \times M$.

1. *Show that the following matrices are also Hermitian:*
   (a) $\sum_{i=1}^{N} A^{(i)}$
   (b) $A^{(i)} \left(A^{(i)}\right)^H$
   (c) $\sum_{i=1}^{N} A^{(i)} \left(A^{(i)}\right)^H$
2. *Assume that all Hermitian matrices $A^{(i)}$, $i = 1, \dots, N$ can be diagonalized by the same unitary matrix Q. Determine tight upper and lower bounds of $\sum_{i=1}^{N} \frac{\underline{x}^H A^{(i)} \underline{x}}{\underline{x}^H \underline{x}}$ in terms of eigenvalues of the matrices $A^{(i)}$.*
3. *Show that $\sum_{i=1}^{N} \left(\frac{\underline{x}^H A^{(i)} \underline{x}}{\underline{x}^H \underline{x}}\right)^2$ is bounded by*

$$\frac{1}{N} \left(\frac{\underline{x}^H \sum_{i=1}^{N} A^{(i)} \underline{x}}{\underline{x}^H \underline{x}}\right)^2 \leq \sum_{i=1}^{N} \left(\frac{\underline{x}^H A^{(i)} \underline{x}}{\underline{x}^H \underline{x}}\right)^2 \leq \sum_{i=1}^{N} \frac{\underline{x}^H A^{(i)} A^{(i),H} \underline{x}}{\underline{x}^H \underline{x}}, \qquad (1)$$

   where $\underline{x}$ is an arbitrary non-zero vector.
   (Hint: use Cauchy-Schwarz.)

Exercise 7.2 (0.5 points)
Consider a square matrix Q with $||Q|| < 1$ and the norm $||\cdot||$ satisfies the submultiplicative property

1. *Proof the following inequality:*

$$||(I-Q)^{-1}|| \leq \frac{1}{1-||Q||}$$

(Hint: Use the Neumann expansion (Theorem 4.2))

2. *Furthermore show*

$$||I-(I-Q)^{-1}|| \leq \frac{||Q||}{1-||Q||}$$

(Hint: Show that $I-(I-Q)^{-1} = -Q(I-Q)^{-1}$)

Now assume A to be nonsingular and E such that $||A^{-1}E|| < 1$.

3. *Show that $A+E$ is nonsingular. (Hint: utilize previous results)*
4. *Let $Q = -A^{-1}E$. Show that Q satisfies:*

$$(A+E)^{-1} = (I-Q)^{-1}A^{-1}$$

5. *Show that $(A+E)^{-1} - A^{-1} = -A^{-1}E(A+E)^{-1}$*
6. *Finally, show that:*

$$||(A+E)^{-1} - A^{-1}|| \leq \frac{||A^{-1}||^2||E||}{1-||Q||} \quad (2)$$

Exercise 7.3 (0.75 points)
Consider the iterative algorithm introduced in Example 4.24 in the lecture notes. In this exercise, we apply the algorithm for calculating the inverse R^{-1} of a Hermitian matrix $R = AA^H$ of dimension $N_R \times N_R$.

1. *Consider the modified algorithm*

$$W_{k+1} = W_k + \mu_k W_k (I - R W_k^H W_k) \tag{3}$$

with bounded step size

$$0 < \mu_k < \frac{2}{\sigma_k^{(max)} \left[1 + \sigma_k^{(max)}\right]}, \tag{4}$$

where $\sigma_k^{(max)}$ denotes the maximum singular value of $W_k A$. Assume that the Singular Value Decomposition (SVD) of $W_k A$ is $W_k A = U \Sigma_k V^H$.

Your task is to analytically show that the modified algorithm as given in Equation (3) converges towards $W_\infty^H W_\infty = R^{-1} = \left(AA^H\right)^{-1}$.
(Hint: multiply Equation (3) with A from the right and show that the singular values in Σ_k converge towards 1, given Equation (4) is satisfied. Use this results to proof the convergence of $W_\infty^H W_\infty$.)

2. *Write a MATLAB program which performs the iterative algorithm to compute the inverse. Use the relative Mean Squared Error (MSE)*

$$MSE_k = \frac{1}{N_R^2} \left\| R W_k^H W_k - I \right\|_F^2 \tag{5}$$

to show the convergence. Your output should be a plot of MSE versus the iteration step. For the step size calculation, use the relation

$$0 < \frac{2}{0.25 + 2\lambda_k^{(max)}} < \frac{2}{\sigma_k^{(max)} \left[1 + \sigma_k^{(max)}\right]}, \tag{6}$$

where $\lambda_k^{(max)} = \left(\sigma_k^{(max)}\right)^2$ denotes the maximum eigenvalue of $W_k R W_k^H$. Generate a suitable matrix for inversion (of dimension $N_R = 5$) by using the `randn` *command in MATLAB. Set $W_0 = I$.*

MATLAB-Exercise 7.1 (1.25 points)

Consider the following signal model:

$$s_k = \sum_{i=1}^{p} a_p e^{j2\pi f_p k} + n_k \tag{7}$$

The signal is a superposition of p sinusoids with amplitudes a_p and frequencies f_p and it is disturbed by additive white Gaussian noise n_k of variance $\sigma_n^2 = 0.1$.

1. *Assume $p = 3$, $f_1 = 0.1$, $f_2 = 0.25$, $f_3 = 0.4$ and $a_1 = \frac{1}{\sqrt{2}}$, $a_2 = \frac{1}{\sqrt{4}}$ and $a_3 = \frac{1}{\sqrt{8}}$. Implement the signal model in Matlab and generate 100 samples. In order to make your results reproducible, use a fixed* seed *to setup the random number generator of Matlab. To generate a random number stream with a fixed seed use the following code:*

   ```
   stream = RandStream('mt19937ar','Seed',1);   (8)
   ```

   When you generate the random noise, handover this stream to the function `randn` *(the Matlab documentation of* `randn` *tells you how to do that).*

2. *From the generated noisy signal-samples, estimate the number of sinusoids p based on an estimate of the autocorrelation matrix of the signal of size 20×20. Do **NOT** use Matlab internal functions like* `xcorr` *or* `cov` *for that purpose. Plot (*`stem`*) the eigenvalues (*`eig`*) of the autocorrelation matrix.*

3. *Estimate the noise power from the appropriate eigenvalues of the autocorrelation matrix and state the estimated value.*

4. *Using the PHD algorithm estimate the frequencies f_p of the sinusoids. Plot your decision function and state the estimated values.*

5. *Using the MUSIC algorithm estimate the frequencies f_p of the sinusoids. Plot your decision function and state the estimated values.*

6. *Repeat the same steps for a signal length of 1000 samples.*