Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 8.1 (0.75 points)

1. *Prove the following equality*

$$e^{A} \otimes e^{B} = e^{A \oplus B}. \tag{1}$$

2. *Prove the following equality*

$$\operatorname{trace}(ABC) = \operatorname{vec}\left(A^{\mathrm{T}}\right)^{\mathrm{T}} (I \otimes B) \operatorname{vec}(C). \tag{2}$$

3. *Consider the following $N \times N$ matrix:*

$$\bar{I}_N = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & 0 & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{bmatrix}$$

Calculate the eigenvalues of

- $\bar{I}_N$
- $I_N \otimes \bar{I}_N$
- $\bar{I}_N \otimes I_N$
- $M_N = I_N \otimes \bar{I}_N + \bar{I}_N \otimes I_N$

Exercise 8.2 (0.75 points)

1. *Consider the following equation:*

$$AX - XB = C, \tag{3}$$

where A,B *and* C *are full-rank known* $n \times n$ *matrices and* X *is to be determined.*

   a) *Show that the equation can be rewritten as*

$$\left((I \otimes A) + (-B^T \otimes I)\right) \underline{x} = \underline{c} \tag{4}$$

   where $\underline{x} = Vec(X)$ *and* $\underline{c} = Vec(C)$.

   b) *Show that the equation has a unique solution if and only if* A *and* B *have no common eigenvalues.*

2. *Show that if* H_n *and* H_m *are Hadamard matrices of order* n *and* m *respectively, then* $H_n \otimes H_m$ *is also a Hadamard matrix. What is its order?*

3. *Let the projection matrix* P_Q *be generated from a unitary matrix* $Q \in \mathbb{C}^{N \times N}$. *Calculate the eigenvalues of* P_Q *and* $I - P_Q$. *Now investigate the matrix* $\tilde{Q}$, *which contains* $M < N$ *columns of* Q. *What are the eigenvalues of* $P_{\tilde{Q}}$*?*

Exercise 8.3 (0.75 points)

1. *Let* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{p\times q}$. *Show that*

$$A \otimes B = (A \otimes I_p)(I_n \otimes B) = (I_m \otimes B)(A \otimes I_q). \tag{5}$$

2. *Now, let* $A =\in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{q\times r}$, $C \in \mathbb{R}^{n\times p}$ *and* $D \in \mathbb{R}^{r\times s}$. *Show that*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{6}$$

(Hint: exploit your previous result.)

3. *Let* A *be of size* $m \times m$ *and* B *be of size* $n \times n$. *Show that*

$$\det(A \otimes B) = \det(A)^n \det(B)^m. \tag{7}$$

4. *Show that* $\operatorname{trace}(A \otimes B) = \operatorname{trace}(A)\operatorname{trace}(B)$.

5. *If* $\underline{x} = [x_1, \dots, x_m]^T$ *is an eigenvector of* A *corresponding to the eigenvalue* α, *and* $\underline{y} = [y_1, \dots, y_n]^T$ *is an eigenvector of* B *corresponding to* β, *then show that* $\underline{z} = [x_1\underline{y}^T, \dots, x_m\underline{y}^T]^T$ *is an eigenvector of* $A \otimes B$ *associated with* $\alpha \cdot \beta$.

Exercise 8.4 (0.75 points)
Define the vec-permutation matrix as

$$P_{mn} \triangleq \begin{bmatrix} I_m \otimes e_{1n}^T \\ I_m \otimes e_{2n}^T \\ \vdots \\ I_m \otimes e_{nn}^T \end{bmatrix} \tag{8}$$

with $P_{mn} \in \mathbb{R}^{mn \times mn}$*, and* $e_{in} \triangleq [0,\ldots,0,1,0,\ldots,0]$ *is a* n*-dimensional vector which has a* 1 *at the* i*-th position and zeros elsewhere. Show that*

1. P_{mn} *can equivalently be expressed as*

$$\sum_{j=1}^{m} \sum_{k=1}^{n} (e_{kn} \otimes e_{jm})(e_{jm} \otimes e_{kn})^T, \tag{9}$$

2. $P_{mn}^T = P_{mn}$*, and*
3. $P_{mn} P_{mn}^T = P_{mn}^T P_{mn} = I_{mn}$.