1 Basic Math, Part 1

Exercise 1.1

Determine the output sequence of a linear time-invariant system, with the following impulse responses h_n and input sequences x_n given (u_n denotes a step function):

1. $x_n = u_n$ *and* $h_n = a^n u_{-n-1}$ *with* $a > 1$.
2. $x_n = u_{n-4}$ *and* $h_n = 2^n u_{-n-1}$.
3. $x_n = u_n$ *and* $h_n = (0{,}5)2^n u_{-n}$.
4. $x_n = u_n - u_{n-10}$ *and* $h_n = 2^n u_{-n-1}$.

Solutions:

1. $y_n = \begin{cases} \frac{a^{n+1}}{a-1} & ; n < 0 \\ \frac{1}{a-1} & ; n \geq 0 \end{cases}.$

2. $y_n = \begin{cases} 2^{n-3} & ; n < 4 \\ 1 & ; n \geq 4 \end{cases}.$

3. $y_n = \begin{cases} 2^{n} & ; n < 1 \\ 1 & ; n \geq 1 \end{cases}.$

4. $y_n = \begin{cases} 2^{n+1} - 2^{n-9} & ; n < 0 \\ 1 - 2^{n-9} & ; 0 \leq n < 10 \\ 0 & ; n \geq 10 \end{cases}.$

Exercise 1.2

Consider the following LTI-system with the frequency response

$$H(e^{j\Omega}) = \frac{1 - e^{j2\Omega}}{1 + 0{,}5e^{-j4\Omega}}, \qquad -\pi < \Omega \leq \pi.$$

Calculate the output sequence y_n for all n, if the input sequence x_n for all n shows the following form:

$$x_n = \sin\left(\frac{\pi n}{4}\right).$$

Solution:

$$y_n = 2\sqrt{2}\sin\left(\frac{\pi(n-1)}{4}\right).$$

Exercise 1.3
Derive the partial fraction expansion of

1.
$$H(z) = \frac{1 - 3z^{-1}}{1 - 1{,}5z^{-1} + 0{,}56z^{-2}}$$

2.
$$H(z) = \frac{10 - 8{,}5z^{-1} + 1{,}35z^{-2}}{1 - 1{,}4z^{-1} + 0{,}45z^{-2}}$$

3.
$$H(z) = \frac{2 - 3z^{-1}}{(1 - 0{,}3z^{-1})^2}$$

4.
$$H(z) = \frac{5 - 6z^{-1}}{(1 - 0{,}3z^{-1})^2(1 - 0{,}4z^{-1})}$$

Solutions:

1.
$$\frac{1 - 3z^{-1}}{1 - 1{,}5z^{-1} + 0{,}56z^{-2}} = \frac{23}{1 - 0{,}7z^{-1}} - \frac{22}{1 - 0{,}8z^{-1}}$$

2.
$$\frac{10 - 8{,}5z^{-1} + 1{,}35z^{-2}}{1 - 1{,}4z^{-1} + 0{,}45z^{-2}} = 3 + \frac{2}{1 - 0{,}5z^{-1}} + \frac{5}{1 - 0{,}9z^{-1}}$$

3.
$$\frac{2 - 3z^{-1}}{(1 - 0{,}3z^{-1})^2} = -\frac{8}{(1 - 0{,}3z^{-1})^2} + \frac{10}{1 - 0{,}3z^{-1}}$$

4.
$$\frac{5 - 6z^{-1}}{(1 - 0{,}3z^{-1})^2(1 - 0{,}4z^{-1})} = \frac{120}{1 - 0{,}3z^{-1}} - \frac{160}{1 - 0{,}4z^{-1}} + \frac{45}{(1 - 0{,}3z^{-1})^2}$$

Exercise 1.4
Find $\limsup_{n\to\infty} a_n$ *and* $\liminf_{n\to\infty} a_n$ *for:*

1. $a_n = \cos\left(\frac{2\pi}{3}n\right)$

2. $a_n = \cos\left(\sqrt{2n}\right)$

3. $a_n = 2 + (-1)^n\left(3 - \frac{2}{n}\right)$

4. $a_n = n^2(-1)^n$.

Solutions:

1.
$$\limsup_{n\to\infty} \cos\left(\frac{2\pi}{3}n\right) = 1$$
$$\liminf_{n\to\infty} \cos\left(\frac{2\pi}{3}n\right) = -\frac{1}{2}$$

2.
$$\limsup_{n\to\infty} \cos\left(\sqrt{2}n\right) = 1$$
$$\liminf_{n\to\infty} \cos\left(\sqrt{2}n\right) = -1$$

3.
$$\limsup_{n\to\infty} 2 + (-1)^n \left(3 - \frac{2}{n}\right) = 5$$
$$\liminf_{n\to\infty} 2 + (-1)^n \left(3 - \frac{2}{n}\right) = -1$$

4.
$$\limsup_{n\to\infty} n^2(-1)^n = \infty$$
$$\liminf_{n\to\infty} n^2(-1)^n = -\infty$$

Exercise 1.5
Calculate for $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$*:*

1. A^2,

2. A^3,

3. $f(A)$,

where $f(x) = 2x^3 - 4x + 5$.

Solutions:

1. $A^2 = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$.

2. $A^3 = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$.

3. $f(A) = \begin{bmatrix} -13 & 52 \\ 104 & -117 \end{bmatrix}$.

Exercise 1.6
Evaluate the inverse of $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$.

Solutions: $A^{-1} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$.

Exercise 1.7
Show the following properties of the Fourier-transform:

$$\begin{aligned} x_e(t) = \frac{1}{2}\left(x(t) + x^*(t)\right) \quad &\Leftrightarrow \quad \Re\{X(j\omega)\} \\ x_o(t) = \frac{1}{2}\left(x(t) - x^*(t)\right) \quad &\Leftrightarrow \quad j\Im\{X(j\omega)\} \\ \Re\{x(t)\} \quad &\Leftrightarrow \quad X_e(j\omega) = \frac{1}{2}\left(X(j\omega) + X^*(-j\omega)\right) \\ j\Im\{x(t)\} \quad &\Leftrightarrow \quad X_o(j\omega) = \frac{1}{2}\left(X(j\omega) - X^*(-j\omega)\right) \end{aligned}$$

Exercise 1.8
Show the properties of Exercise 1.7 for the time-discreet Fourier-transform.
