Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 3.1 (0.5 points)
Show that the Galois Field (GF) 3 fulfills all field properties.

The tables for the '+' and '∗' operation are given as

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

∗	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Terms: Galois field, field properties.

Exercise 3.2 (0.5 points)
Consider the sequence

$$x_{n+1} = \frac{x_n + \frac{A}{x_n}}{2} \tag{1}$$

with

$$n = 0,1,2..., \qquad x_0 \neq 0, \qquad A > 0$$

1. (a) Consider $A = 9$. Does the sequence converge in $\mathbb{Q}$? Does it converge in $\mathbb{R}$? If yes, to which value does it converge? Is it a Cauchy sequence in $\mathbb{Q}$/ in $\mathbb{R}$?

   (b) Consider $A = 2$. Does the sequence converge in $\mathbb{Q}$? Does it converge in $\mathbb{R}$? If yes, to which value does it converge? Is it a Cauchy sequence in $\mathbb{Q}$/ in $\mathbb{R}$?

   Do not only examine convergence, but also if the definition for Cauchy sequences holds:

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m,n \geq N : |x_m - x_n| < \epsilon$$

Do all elements of the sequence lie in the corresponding set?

Hint: Try to find a lower and upper bound for x_{n+1}; these bounds can also be used to show that it is a Cauchy Sequence, if the sequence converges; if the sequence does not converge, the proof is more tricky - don't worry too much, if you can't do it...

The sequence introduced in Equation (1) can be used to approximate the square root of a number. It is only a special case of a different method to find the roots of an arbitrary function. The general sequence is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

with

$$n = 0,1,2...$$

2. (a) Find the sequence to approximate the cubic root of a number by using Equation (2) applied at $f(x) = x^3 - V, \quad V > 0, \quad x_0 > 0$.
   (b) Show that the sequence converges to $\sqrt[3]{V}$.
   (c) The sequence does not converge for some $x_0 < 0$. Find such a value.

Terms: Cauchy sequence, convergence.

Exercise 3.3 (0.5 points)
Consider the phase vector

$$\mathbf{v} = \begin{bmatrix} \exp(j\phi_1) \\ \exp(j\phi_2) \end{bmatrix}, \quad \phi_1, \phi_2 \in \mathbb{R}$$

and a vector

$$\mathbf{p} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad a, b \in \mathbb{C}.$$

1. Assume given $\mathbf{p}$ and variable $\mathbf{v}$. Which achievable points c can be reached by the inner product $c = \langle \mathbf{v},\mathbf{p} \rangle$? Draw the region in the complex plane.

2. Given an achievable point c and a fixed $\mathbf{v}$, how do you find $\mathbf{p}$?
   How many solutions for $\mathbf{p}$ do exist?

3. Given an achievable point c and a fixed $\mathbf{p}$, how do you find $\mathbf{v}$?
   Hint: the law of cosines may be helpful.
   How many solutions for $\mathbf{v}$ do exist?

Terms: inner product, law of cosines, complex numbers.

MATLAB-Exercise 3.1 (1 points)

In this exercise, we want to visualize our findings from exercise 3.2.

1. Reconsider Equation (1). Now let $A = 64$ and $x_0 = 1$. Plot the whole sequence for $n = 1...10$. For each iteration-step, plot x_n and $y_n = \frac{A}{x_n}$ as the two sides of a rectangle with the area A.

2. Now, reconsider Equation (2) for the approximation of the cubic root of A. Now let $A = 27$ and $x_0 = 10$. Plot the function $f(x) = x^3 - A$ and for each iteration plot the tangent to the function $f(x)$ and check where it intersects with the x-axis.