Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 4.1 (0.5 points)
Show that the following expressions define norms on $\mathbb{R}^n$, where $\underline{x} = [x_1, x_2, \ldots, x_n]^T$:

- $\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$,
- $\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$,
- $\|\underline{x}\|_\infty = \max_{1 \le i \le n} |x_i|$.

Do you know a metric that is not a norm?
Terms: norm properties, triangle inequality, Cauchy-Schwarz inequality.

Exercise 4.2 (0.5 points)
Consider the system shown in Figure 4.1. Note that $\underline{x}_N = [x_0, \ldots, x_k, \ldots, x_N]^T$.
H and G are defined as

$$H(z) = 1 + 0.5z^{-1}; \qquad G(x) = \frac{\alpha}{1 + x^2}$$

with $z_k = G(g_k)g_k$ in a real-valued vector space.

1. Calculate the gain of H and G according to the l_∞-norm.
2. Find α so that the closed-loop system is l_∞-stable.
3. Let x_k be the input signal and y_k the output signal. Specify the robustness of the system in presence of additive noise v_k.

   Hint: Use the following formula for robustness:

   $$\gamma_R = \sup_{v \in I_h(0,N)} \frac{\sqrt{\sum_{i=0}^{N} y_i^2}}{\gamma_v \sqrt{\sum_{i=0}^{N} v_i^2}}.$$

   Find a suitable expression for $\underline{y}_N$ that includes $\underline{v}_N$, G_N, H_N; then use l_2-norm, submultiplicative property and triangle inequality to find an expression for γ_R. Note that $\underline{x}_N$ is of no interest regarding our robustness definition.

Hints to handle operator $G(x)$:

- *We can write $\underline{z}_N = G_N \underline{g}_N$, where G_N is a diagonal matrix. How do the diagonal elements look like?*
- *It holds that $\left\| G_N \underline{g}_N \right\|_p \leq \|G_N\|_p \left\| \underline{g}_N \right\|_p$, $p = 2, \infty$.*
- *To compute the gain according to Equation (2.2) from the lecture notes, assume that $\beta = 0$ in our case (which is a valid choice for β).*
- *Note that G_N operates instantaneous on $\underline{g}_N$ and is memoryless. This can be used to facilitate the computation of the induced norm $\|G_N\|_p$. Can you show that $\|G_N\|_p = |\alpha|$?*

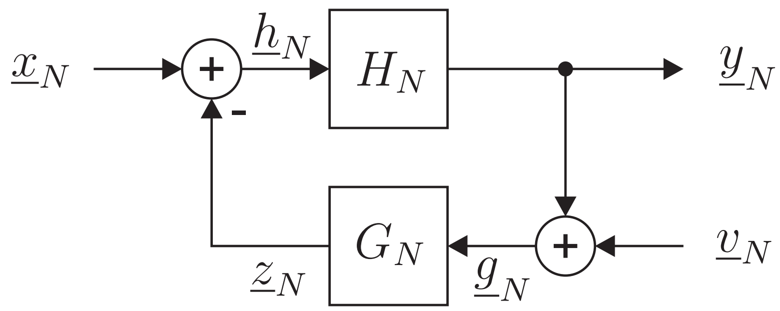

Figure 4.1: Systems H and G.

Terms: stability, small gain theorem, robustness.

Exercise 4.3 (0.5 points)
Consider the three signals $s_1(t)$, $s_2(t)$ and $s_3(t)$ of Figure 4.2.

1. Evaluate $\|s_1(t)\|_2$, $\|s_2(t)\|_2$ and $\|s_3(t)\|_2$
2. Derive an orthonormal basis for the space spanned by $s_1(t)$, $s_2(t)$ and $s_3(t)$. Use the Gram-Schmidt orthogonalization method and start with $s_1(t)$. Sketch the evaluated basis functions. Which dimension has the signal space?
3. Represent the two signals illustrated in Figure 4.3 in terms of the obtained basis (i.e., by specifying the coefficients).
4. Calculate $\|a(t) - b(t)\|_2$.

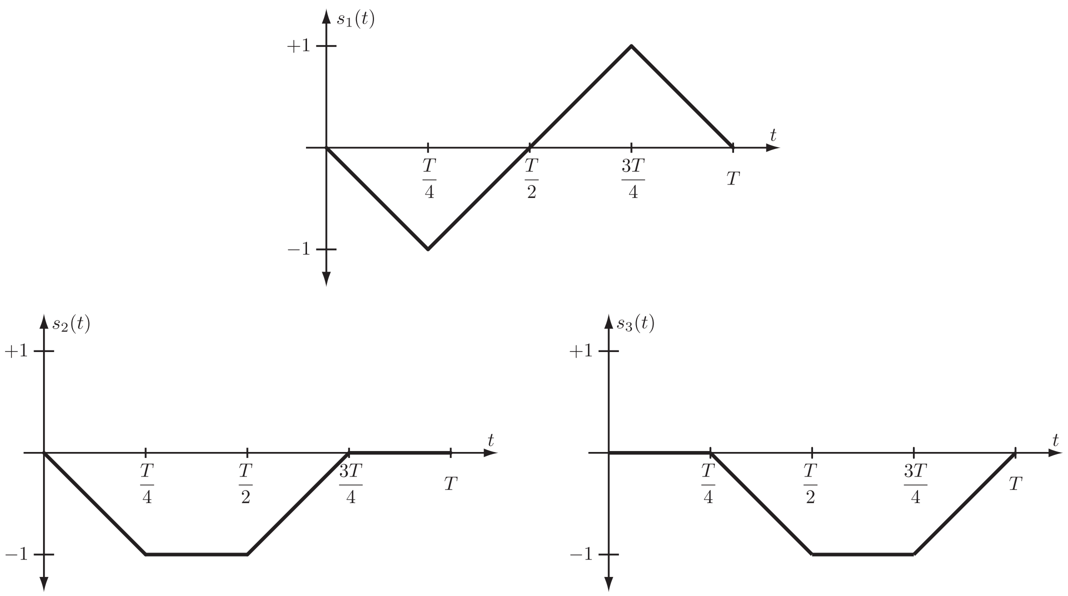

Figure 4.2: Signals $s_1(t)$, $s_2(t)$ and $s_3(t)$.

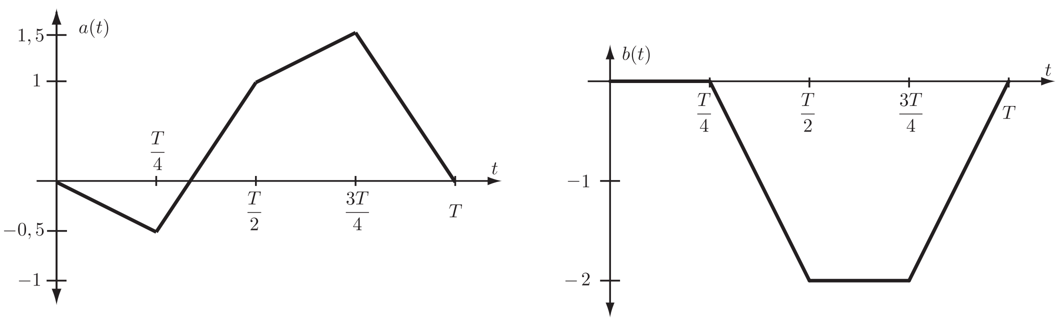

Figure 4.3: Signals $a(t)$ and $b(t)$.

MATLAB-Exercise 4.1 (1 point)
Consider the encrypted image in Figure 4.4, which can be found on the lecture's hompage, as a .mat-file (`data.mat`). It contains four different images, which are encoded by utilizing the orthogonal spreading sequences of the UMTS standard TS25.213 (http://www.3gpp.org).
Hint: the standard refers to these sequences as "Orthogonal Variable Spreading Factor (OVSF) codes".

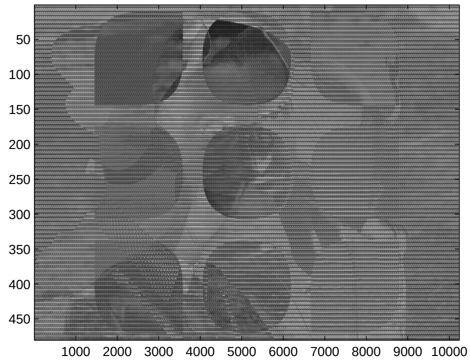

Figure 4.4: Encrypted pictures.

The following is known:

- Size of the pictures: 640 x 480, grayscale
- Spreading code sequences (length 16) (used in a cyclical way for consecutive image lines):
  - Picture 1: 3, 5, 8, 12, 16, 2
  - Picture 2: 1, 6, 9, 15, 10, 7
  - Picture 3: 7, 12, 14, 5, 4, 3
  - Picture 4: 2, 7, 1, 15, 11, 8

Write a MATLAB script which is able to decode and separate the four encoded and superposed pictures and displays them (grayscale). Use the command
`image(uint8(P),'CDataMapping','scaled'); colormap(gray)`,
where `P` represents your MATLAB matrix containing the decoded image data. Then answer the following questions:

1. Which animal is depicted? "Bonus": What famous Austrian artist is depicted?
2. What did the encoders of the pictures do wrong, i.e., why cannot all pictures be decoded correctly?

> *Hint: The code used for encoding the pictures indexes the sequence-vector with* `seq_vec(1+mod(x,length(seq_vec)))` *with* `x` *starting at 1 - consider this for your decision with which sequence to start for the decoding!*

Terms: Spreading sequences, encryption.