Guidelines

- Write your name and matriculation number on each sheet of paper.

- Only clearly readable exercise-elaborations are evaluated.

- Results have to be provided together with an evident way of calculation.

- Keep textual answers short and concise. Lengthy or vague statements won't gain points.

Exercise 6.1 (0.5 points)
Consider the plane $\mathcal{A}$, defined as

$$\mathcal{A} : \underline{y} = \alpha \underline{a} + \beta \underline{b}$$

with

$$\underline{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

and a vector $\underline{x}$

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

1. (a) What is the condition for the minimal error $\underline{e} = \underline{x} - \hat{\underline{x}}$ with $\hat{\underline{x}} \in \mathcal{A}$ (minimal in the sense of the l2-norm)?

   (b) Define a system of equations to determine α and β that correspond to this minimal error. How many equations and how many unknowns are there?

Now rewrite the definition of the plane $\mathcal{A}$ in matrix-vector form:

$$\mathcal{A} : \underline{y} = A\underline{\lambda}$$

with

$$A = \begin{bmatrix} \underline{a}, & \underline{b} \end{bmatrix}, \quad \underline{\lambda} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

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2. (a) Reformulate $\underline{e} = \underline{x} - \hat{\underline{x}}$ in matrix-vector notation and find $\underline{\lambda}$ for the given values.

    (b) Find a matrix C so that $\hat{\underline{x}} = C\underline{x}$ and express it in terms of A. Calculate $\hat{\underline{x}}$ for the given values.

    (c) Find a matrix D so that $\underline{e} = D\underline{x}$ and express it in terms of A. Calculate $\underline{e}$ for the given values.

3. (a) Show that C is a projection matrix.

    (b) Show that D is a projection matrix.

    (c) Show that C and D perform projections on orthogonal subspaces.

Terms: Pseudo inverse, projection.

Exercise 6.2 (0.5 points)
Reconsider the example from Exercise 3 where we had a phase vector

$$\underline{v} = \begin{bmatrix} \exp(j\phi_1) \\ \exp(j\phi_2) \end{bmatrix}, \quad \phi_1, \phi_2 \in \mathbb{R}$$

and a vector

$$\underline{p} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad a, b \in \mathbb{C}.$$

1. "Given an achievable point c and a fixed $\underline{v}$, how do you find $\underline{p}$?"
   Formulate this problem in terms of a matrix-vector multiplication by considering real and imaginary parts separately:

   $$\underline{c}_{RI} = \begin{bmatrix} c_R \\ c_I \end{bmatrix} = V \underline{p}_{RI},$$

   where $c_R = \mathrm{Re}\{c\}$, $c_I = \mathrm{Im}\{c\}$. Determine V and $\underline{p}_{RI}$ and find the least squares solution for $\underline{p}_{RI}$ (attention: do we need the left or right pseudoinverse of V?).
   Hint: use Euler's formula and then separate real and imaginary part.
   What can you say about the so obtained solution for $\underline{p}$?

2. Considering the phase vector $\underline{v}(\underline{\Phi}) \in \mathbb{C}^2, \underline{\Phi} \in \mathbb{R}^2$, show on an example that:

   - Two linearly *independent* vectors $\underline{\Phi}_a$ and $\underline{\Phi}_b$ do not necessarily lead to linearly independent phase vectors $\underline{v}(\underline{\Phi}_a)$ and $\underline{v}(\underline{\Phi}_b)$.
   - Two linearly *dependent* vectors $\underline{\Phi}_a$ and $\underline{\Phi}_b$ do not necessarily lead to linearly dependent phase vectors $\underline{v}(\underline{\Phi}_a)$ and $\underline{v}(\underline{\Phi}_b)$.

Let us increase the problem dimension to m, i.e., $\underline{p} = [p_1, ..., p_m]^T$ and $\underline{v} = [\exp(j\phi_1), ..., \exp(j\phi_m)]^T$.

3. Given a vector $\underline{p} \in \mathbb{C}^m$, find the phase vector $\underline{v} \in \mathbb{C}^m$ that approximates it in the least squares sense (minimizes error $\underline{e} = \underline{p} - \underline{v}$).
   Hint: illustrate the vectors in the complex plane with $m = 1$.

Terms: inner product, least squares, minimum norm solution, Euler's formula, linear dependency.

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Exercise 6.3 (0.5 points)

1. Show that the nullspace of an operator A is a linear vector space.

2. Prove the remaining parts of Theorem 4.7 in the lecture notes.

3. Consider the space of polynomials of degree $\leq n$: $\mathcal{P}_n = \{c_0 + c_1 x + \ldots + c_n x^n\}$. Find the range and null space of the differentiation operator D : $D(p) = \frac{d}{dx} p(x)$, $p(x) \in \mathcal{P}_n$. State the dimension of these two spaces. Do the same for the integration operator $T : T(p) = \int_0^x p(t) dt$.

4.) Consider a linear map $T : V \to W$ between two vector spaces.

   a) Suppose that $N(T) = \{0\}$ and that $(\underline{v}_1, \cdots, \underline{v}_n)$ are linearly independent in V. Show that $(T\underline{v}_1, \cdots, T\underline{v}_n)$ is linearly independent in W.

   b) Suppose $V = \mathbb{R}^3$ and $W = \mathbb{R}^4$. If there are at least two independent vectors in $N(T)$, what are the possible values for the dimension of $R(T)$?

Terms: Pseudo inverse, projection.

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MATLAB-Exercise 6.1 (1 point)
In this Matlab exercise we investigate the Haar-wavelet transform. We will see that most of the orthogonal expansion coefficients are close to or equal to zero and that an image can therefore be effectively described by a few coefficients only. This enables a lossless compressed description of the image, as well as lossy compression by means of approximating the image with the largest coefficients only.

Let us assume a vector $\underline{x}$ of even length N. The one-dimensional Haar-wavelet transform $\underline{x}_t$ of the vector $\underline{x}$ is calculated with the following two operations:

- The first $\frac{N}{2}$ coefficients (low frequency coefficients) are computed as the normalized average between each odd entry of the vector $\underline{x}$ and the consecutive even entry:

$$\underline{x}_t(i) = \frac{1}{\sqrt{2}} \left(\underline{x}(2i-1) + \underline{x}(2i) \right), \quad i = 1, \dots, \frac{N}{2} \tag{1}$$

- The last $\frac{N}{2}$ coefficients (high frequency coefficients) are given by the normalized differences between consecutive vector entries:

$$\underline{x}_t \left(\frac{N}{2} + i \right) = \frac{1}{\sqrt{2}} \left(\underline{x}(2i-1) - \underline{x}(2i) \right), \quad i = 1, \dots, \frac{N}{2} \tag{2}$$

1. Assume $N = 4$ and rewrite these operations as inner products with appropriate basis vectors. Verify that the so obtained vectors form an orthonormal basis for the space.

2. How is this transformation inverted?

For two-dimensional pictures X, the tranformation is very similar:

- First the 1D wavelet transform is applied to each column of the picture, to obtain the vertical transform $X_t^{(v)}$.

- Afterwards, each row of the vertical transform $X_t^{(v)}$ is additionally horizontally 1D wavelet transformed (the vertical/horizontal order can also be reversed). Graphically this can be represented as shown in Figure 6.1.

3. Implement functions `my_haar` and `my_inv_haar` in Matlab that realize the Haar-wavelet transform and its inverse. Apply the transformation and its inverse to the picture `pic` that can be found in the Matlab file `picture_small.mat` on the course homepage (see current exercise). Verify that, in accordance with Parseval's theorem, the energy of the image and its transform is the same.
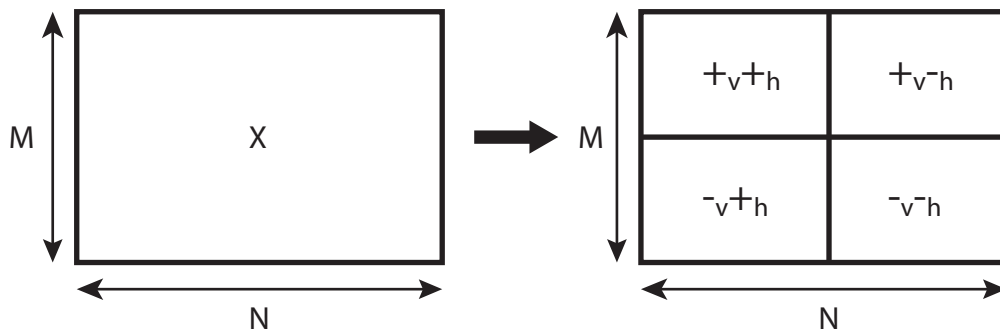

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Figure 6.1: Graphical represenation of the Haar-wavelet transform

4. Plot the original image, its transform and the back-transformed image, using the function `image` and the colormap defined in `colorm` in the file `picture_small.mat`. (Be careful to renormalize your images such that the color dynamic range of the colormap is fully utilized).

5. Furthermore, sort the absolute values of all coefficients of the wavelet transform in increasing order and plot them. How many coefficients are equal to zero?

To further reduce the number of non-zero transform coefficients, this wavelet transformation can be applied recursively. This is achieved by applying the same transformation to the low-frequency coefficients (the $+_v+_h$ part in Figure 6.1) in a recursive way, thereby reducing the size of this part in each step, until only one low-frequency coefficient is left.

6. Implement this recursive transformation. How often can the recursion be applied to the image provided in `pic`?

7. Again plot the sorted absolute values of the transform coefficients, obtained after the maximum number of possible recursions. How many coefficients are now equal to zero?

8. Implement the inverse of the recursive transformation as well (this is just the inverse transformation recursively applied to the corresponding part of the wavelet coefficients). Compute the error energy between the original and the back-transformed image. Why is it not exactly equal to zero?

As you see from the absolute values of the wavelet-transform coefficients, most energy is contained in only a few coefficients. This suggests to approximate/compress the image by discarding (setting equal to zero) the coefficients that are close to zero.

9. Now we approximate our image by keeping only as much of the largest transform coefficients as needed to conserve 90%, 99% and 99.9% of the

total image energy. How many coefficients do you need to retain in order to obtain these values? How does the number of recursions of the transformation influence this value? Plot the number of retained coefficients versus the recursion order in a semi-logarithmic plot (`semilogy`) for the three percentages given above. To what SNRs (total image energy divided by approximation-error energy) do these energy percentages correspond to?

10. Plot the approximated pictures for 90% and 99.9% of conserved energy and the two recursion orders 1 and 8. Which approximations do you personally prefer? Is the SNR therefore an appropriate measure of the quality of the approximation?